A INSUSTENTAVEL LEVEZA DOS CONCEITOS ENTROPICOS E SUAS APLICACOES EM ECONOFISICA

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J.W. GIBBS

Elementary Principles in Statistical Mechanics - Developed with Especial Reference to the Rational Foundation of Thermodynamics

C. Scribner's Sons, New York, 1902; Yale University Press, New Haven, 1981), page 35

> *In treating of the canonical distribution, we shall always suppose the multiple integral in equation (92)* [the partition function, as we call it nowadays] *to have a finite valued, as otherwise the coefficient of probability vanishes, and the law of distribution becomes illusory. This will exclude certain cases, but not such apparently, as will affect the value of our results with respect to their bearing on thermodynamics. It will exclude, for instance, cases in which the system or parts of it can be distributed in unlimited space [...]. It also excludes many cases in which the energy can decrease without limit, as when the system contains material points which attract one another inversely as the squares of their distances. [...]. For the purposes of a general discussion, it is sufficient to call attention to the assumption implicitly involved in the formula (92).*

Enrico FERMI *Thermodynamics* (Dover, 1936)

The entropy of a system composed of several parts is very often equal to the sum of the entropies of all the parts. This is true if the energy of the system is the sum of the energies of all the parts and if the work performed by the system during a transformation is equal to the sum of the amounts of work performed by all the parts. Notice that these conditions are not quite obvious and that in some cases they may not be fulfilled. Thus, for example, in the case of a system composed of two homogeneous substances, it will be possible to express the energy as the sum of the energies of the two substances only if we can neglect the surface energy of the two substances where they are in contact. The surface energy can generally be neglected only if the two substances are not very finely subdivided; otherwise, it can play a considerable role.

ENTROPIC FORMS

	$p_i = \frac{1}{W} \quad (\forall i)$ equiprobability	$\forall p_i \; (0 \le p_i \le 1)$ $\left(\sum_{i=1}^{W} p_i = 1 \right)$
***BG* entropy** **(*q* =*1*)**	$k \, ln \, W$	$-k \sum_{i=1}^{W} p_i \ln p_i$
Entropy *Sq* **(*q real*)**	$k \frac{W^{1-q} - 1}{1-q}$	$k \frac{1 - \sum_{i=1}^{W} p_i^q}{q-1}$

Concave

Extensive

Lesche-stable

Finite entropy production per unit time

Pesin-like identity (with largest entropy production)

Composable

Topsoe-factorizable

Possible generalization of
Boltzmann-Gibbs statistical mechanics

[C.T., J. Stat. Phys. **52**, 479 (1988)]

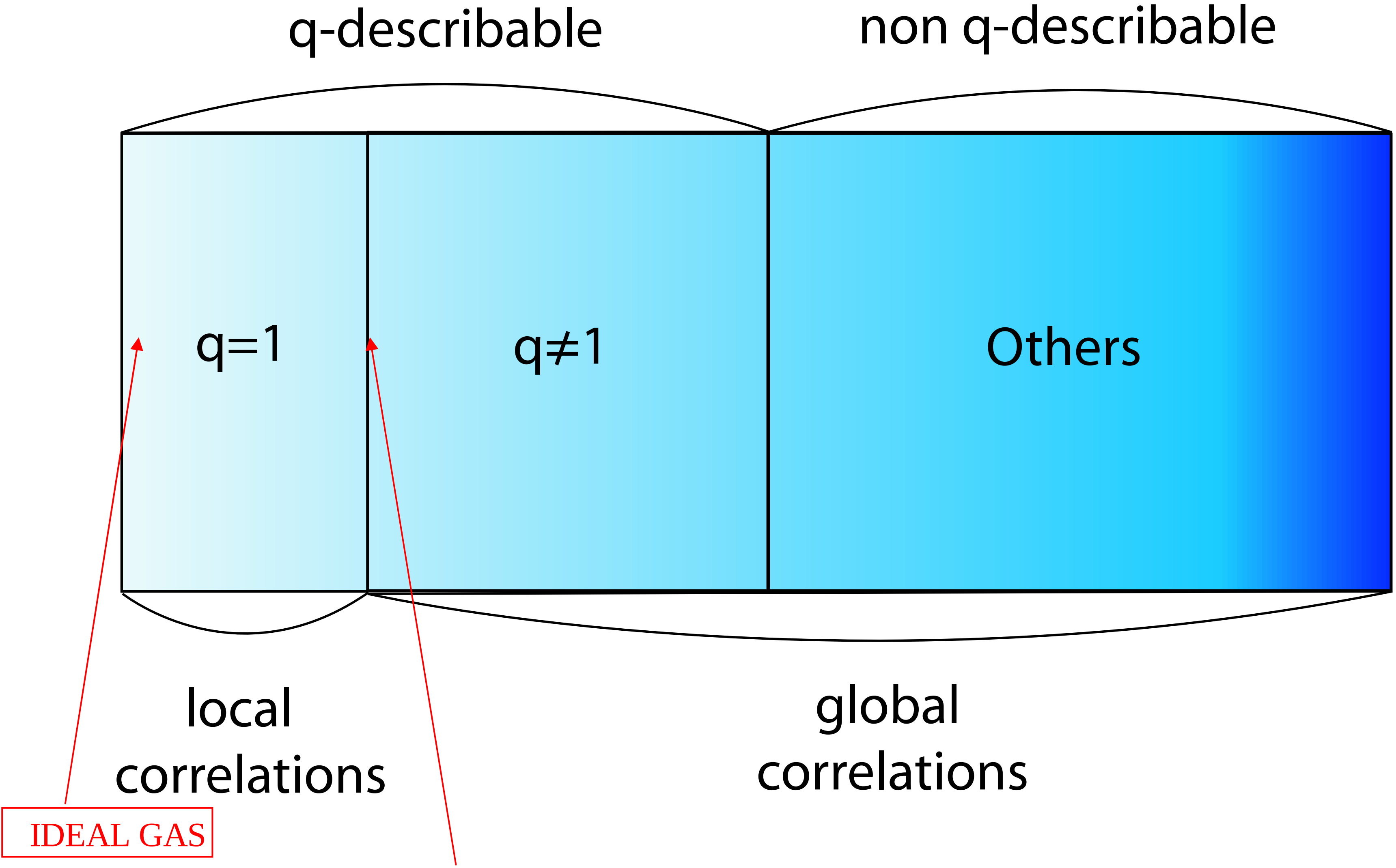
DEFINITION $(q-logarithm)$:

$$\ln_q x \equiv \frac{x^{1-q}-1}{1-q} \quad (x>0)$$

$$\ln_1 x = \ln x$$

Hence, the entropies can be rewritten:

	equal probabilities	*generic probabilities*
BG entropy $(q=1)$	$k \ln W$	$k \sum_{i=1}^{W} p_i \ln \frac{1}{p_i}$
entropy S_q $(q \in R)$	$k \ln_q W$	$k \sum_{i=1}^{W} p_i \ln_q \frac{1}{p_i}$

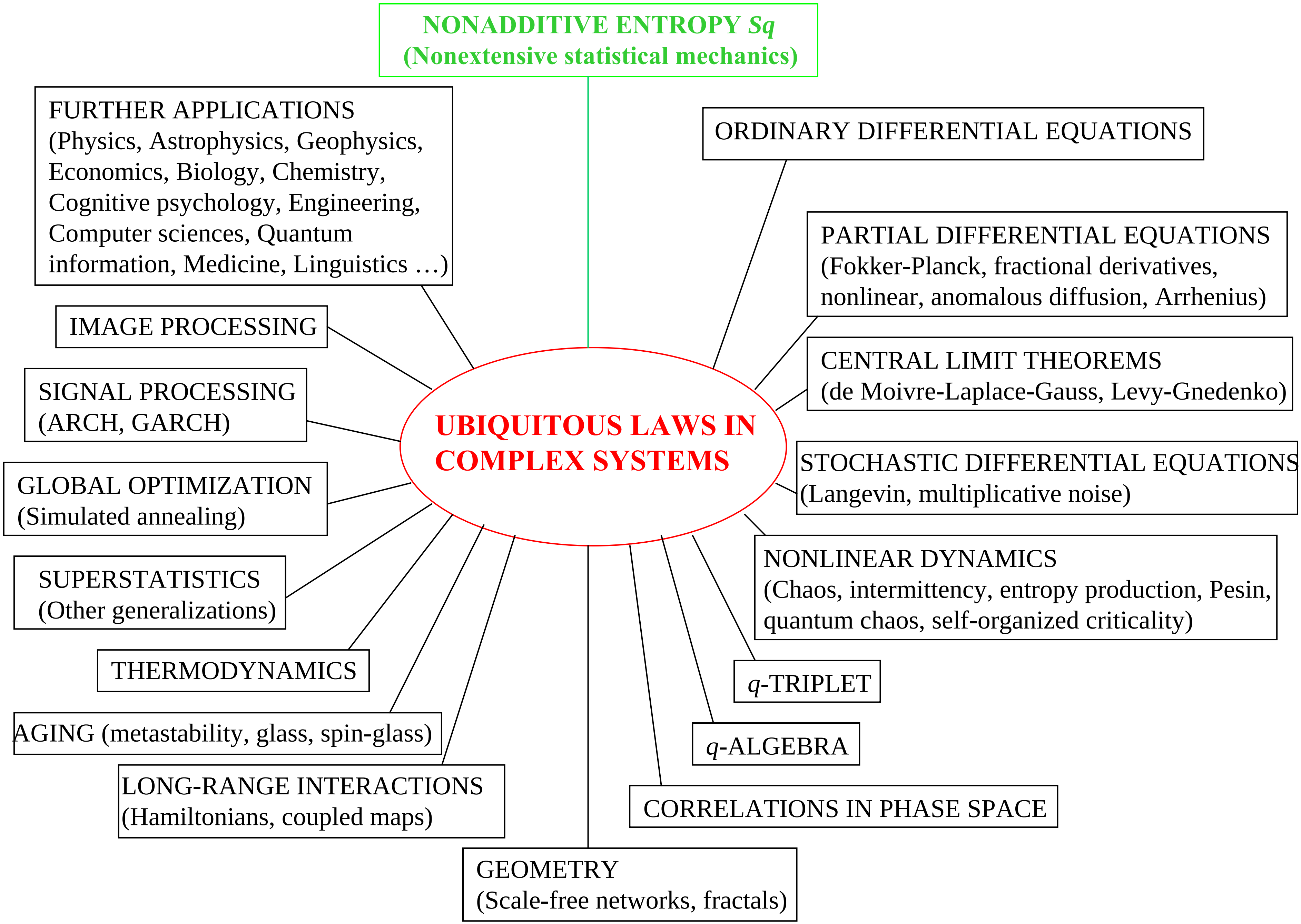


CRITICAL PHENOMENA

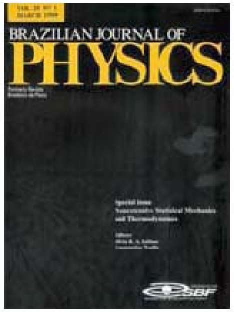
$$q = \frac{1+\delta}{2}$$ (A. Robledo, Mol Phys 103 (2005) 3025)

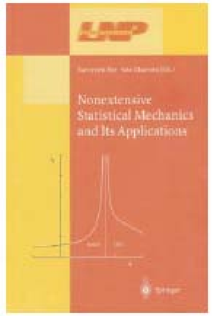
$$q = \frac{\sqrt{9+2c^2}-3}{c}$$ (F. Caruso and C. T., 2006)

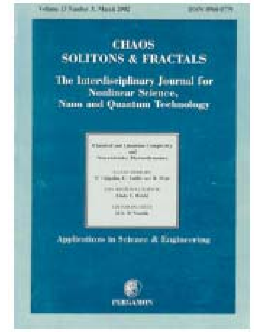
C.T., M. Gell-Mann and Y. Sato
Europhysics News **36** (6), 186
(European Physical Society, 2005)

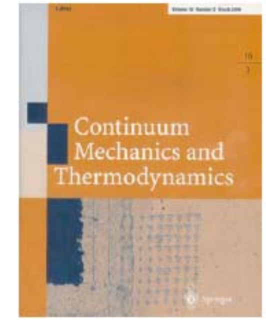

NONADDITIVE ENTROPY Sq
(Nonextensive statistical mechanics)
FURTHER APPLICATIONS (Physics, Astrophysics, Geophysics, Economics, Biology, Chemistry, Cognitive psychology, Engineering, Computer sciences, Quantum information, Medicine, Linguistics …)
ORDINARY DIFFERENTIAL EQUATIONS
PARTIAL DIFFERENTIAL EQUATIONS (Fokker-Planck, fractional derivatives, nonlinear, anomalous diffusion, Arrhenius)
IMAGE PROCESSING
CENTRAL LIMIT THEOREMS (de Moivre-Laplace-Gauss, Levy-Gnedenko)
SIGNAL PROCESSING (ARCH, GARCH)
UBIQUITOUS LAWS IN COMPLEX SYSTEMS
STOCHASTIC DIFFERENTIAL EQUATIONS (Langevin, multiplicative noise)
GLOBAL OPTIMIZATION (Simulated annealing)
NONLINEAR DYNAMICS (Chaos, intermittency, entropy production, Pesin, quantum chaos, self-organized criticality)
SUPERSTATISTICS (Other generalizations)
THERMODYNAMICS
q-TRIPLET
AGING (metastability, glass, spin-glass)
q-ALGEBRA
LONG-RANGE INTERACTIONS (Hamiltonians, coupled maps)
CORRELATIONS IN PHASE SPACE
GEOMETRY (Scale-free networks, fractals)

NONEXTENSIVE STATISTICAL MECHANICS AND THERMODYNAMICS









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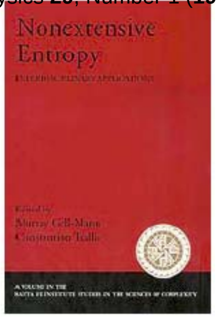

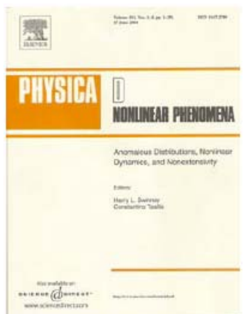

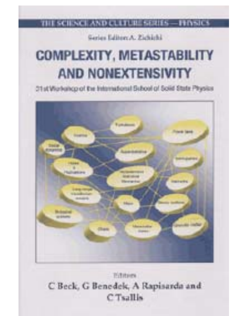

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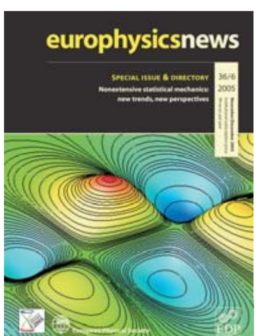

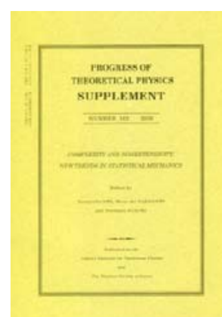

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http://tsallis.cat.cbpf.br/biblio.htm

2,297 articles (done by 1,698 scientists from 60 countries)

[8 November 2007]

$$S_q(N,t)\ versus\ \ N$$

EXTENSIVITY OF THE NONADDITIVE ENTROPY S_q

MATHEMATICAL REALIZATIONS:

Strongly correlated many-variable equal-probability discrete systems

C. T., in *Nonextensive Entropy - Interdisciplinary Applications*,
eds. M. Gell-Mann and C. T. (Oxford University Press, New York, 2004), page 1

Strongly correlated binary variables

C. T., M. Gell-Mann and Y. Sato, Proc Natl Acad Sc (USA) 102, 15377 (2005)

C. T., M. Gell-Mann and Y. Sato, Europhysics News 36, 186 (2005)

Y. Sato and C. Tsallis, in *Complexity: An unifying direction in science*,
eds. T. Bountis, G. Casati and I. Procaccia , Int J Bifurcation Chaos 16, 1727 (2006)

J.A. Marsh, M.A. Fuentes, L.G. Moyano and C. T., Physica A 372, 183 (2006)

PHYSICAL REALIZATIONS (quantum entanglement):

One-dimensional spin $1/2$ XY ferromagnet with transverse magnetic field at $T=0$

F. Caruso and C. T., cond-mat/0612032, in *Complexity, Metastability and Nonextensivity*,
eds. S. Abe, H.J. Herrmann, P. Quarati, A. Rapisarda and C. T.
American Institute of Physics Conference Proceedigs 965 (Dec 2007), in press

Two-dimensional bosonic system of coupled harmonic oscillators at $T=0$

F. Caruso and C. T., preprint (2007)

HYBRID PASCAL - LEIBNITZ TRIANGLE

$(N = 0)$	$1 \times \frac{1}{1}$
$(N = 1)$	$1 \times \frac{1}{2} \quad 1 \times \frac{1}{2}$
$(N = 2)$	$1 \times \frac{1}{3} \quad 2 \times \frac{1}{6} \quad 1 \times \frac{1}{3}$
$(N = 3)$	$1 \times \frac{1}{4} \quad 3 \times \frac{1}{12} \quad 3 \times \frac{1}{12} \quad 1 \times \frac{1}{4}$
$(N = 4)$	$1 \times \frac{1}{5} \quad 4 \times \frac{1}{20} \quad 6 \times \frac{1}{30} \quad 4 \times \frac{1}{20} \quad 1 \times \frac{1}{5}$
$(N = 5)$	$1 \times \frac{1}{6} \quad 5 \times \frac{1}{30} \quad 10 \times \frac{1}{60} \quad 10 \times \frac{1}{60} \quad 5 \times \frac{1}{30} \quad 1 \times \frac{1}{6}$

$$\Sigma = 1 \quad (\forall \; N)$$

Blaise **Pascal** (1623-1662)

Gottfried Wilhelm **Leibnitz** (1646-1716)

Daniel **Bernoulli** (1700-1782)

$(N=2)$

B / A	1	2	
1	$p^2+\kappa$	$p(1-p)-\kappa$	p
2	$p(1-p)-\kappa$	$(1-p)^2+\kappa$	1- p
	p	1- p	1

EQUIVALENTLY:

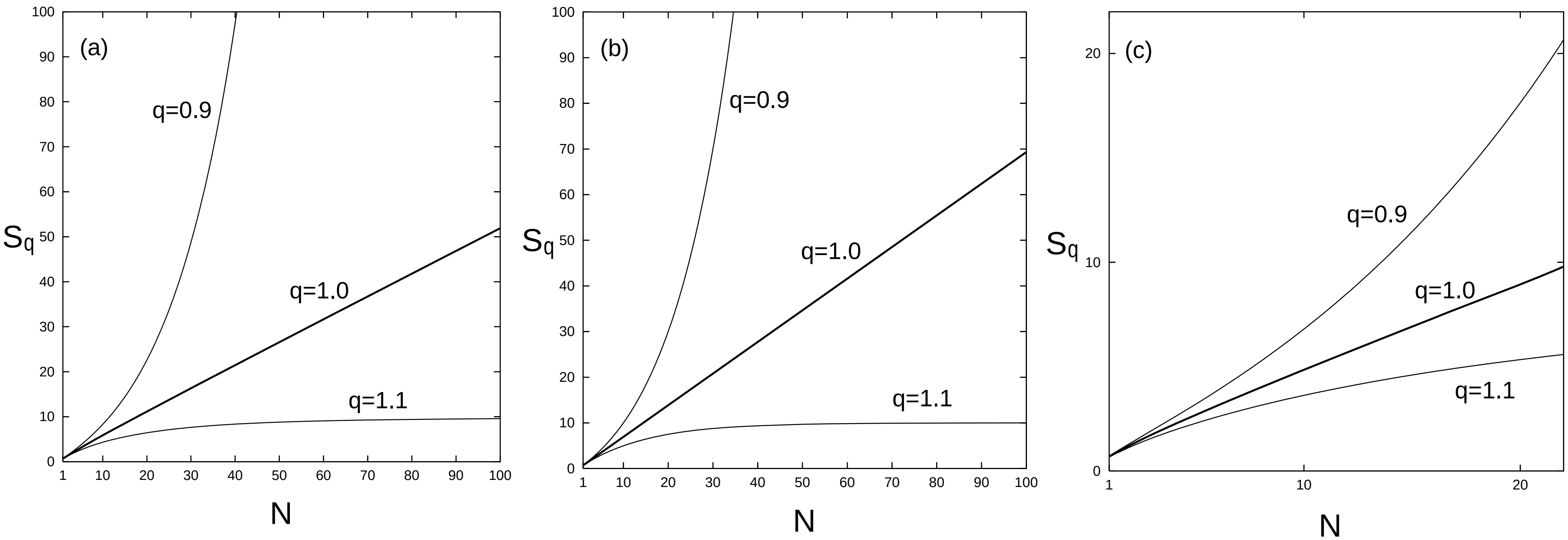
$(N=0)$ 1×1

$(N=1)$ $1\times p$ $1\times(1-p)$

$(N=2)$ $1\times[p^2+\kappa]$ $2\times[p(1-p)-\kappa]$ $1\times[(1-p)^2+\kappa]$

$q = 1$ SYSTEMS

i.e., such that $S_1(N) \propto N \quad (N \to \infty)$



Leibnitz triangle

$$\left(p_{N,0} = \frac{1}{N+1} \right)$$

N independent coins

$$\left(\begin{array}{l} p_{N,0} = p^{N} \\ \\ with\ p = 1/2 \end{array} \right)$$

Stretched exponential

$$\left(\begin{array}{l} p_{N.0} = p^{N^{\alpha}} \\ \\ with\ p = \alpha = 1/2 \end{array} \right)$$

(All three examples **strictly** satisfy the Leibnitz rule)

C.T., M. Gell-Mann and Y. Sato
Proc Natl Acad Sc USA **102**, 15377 (2005)

Asymptotically scale-invariant (d=2)

$$
\begin{array}{lccccccccc}
(N=0) & & & & & 1 & & & & \\
(N=1) & & & & 1/2 & & 1/2 & & & \\
(N=2) & & & 1/3 & & 1/6 & & 1/3 & & \\
(N=3) & & 3/8 & & 5/48 & & 5/48 & & 0 & \\
(N=4) & 2/5 & & 3/40 & & 1/20 & & 0 & & 0
\end{array}
$$

d+1

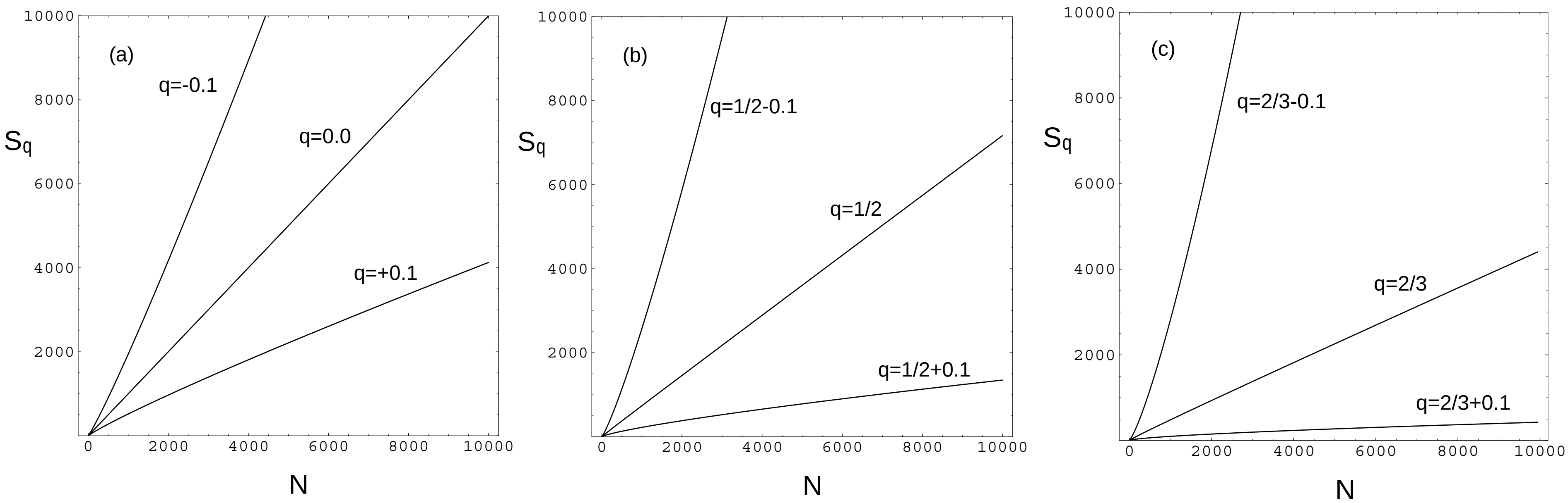
(It **asymptotically** satisfies the Leibnitz rule)

C.T., M. Gell-Mann and Y. Sato
Proc Natl Acad Sc USA **102**, 15377 (2005)

$q \neq 1$ SYSTEMS

i.e., such that $S_q(N) \propto N \quad (N \to \infty)$

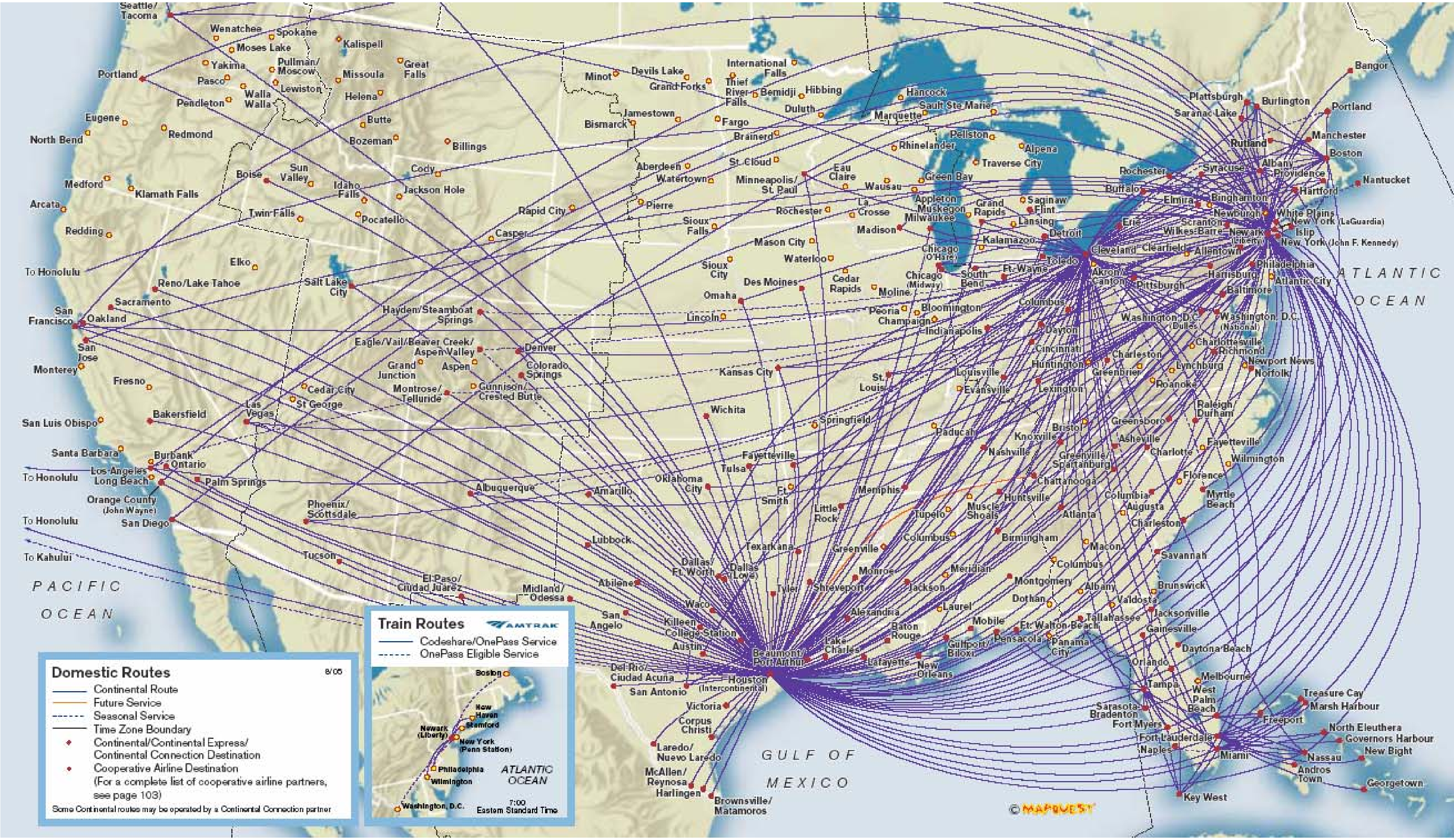
(d =1) (d = 2) (d = 3)



$$q = 1 - \frac{1}{d}$$

(All three examples **asymptotically** satisfy the Leibnitz rule)

C.T., M. Gell-Mann and Y. Sato
Proc Natl Acad Sc USA **102**, 15377 (2005)



Continental Airlines

If A and B are independent,

i.e., if $p_{ij}^{A+B} = p_i^{A}\, p_j^{B}$,

then

$$S_{BG}(A+B) = S_{BG}(A) + S_{BG}(B)$$

whereas

$$S_q(A+B) = S_q(A) + S_q(B) + \frac{1-q}{k_B}\, S_q(A)\, S_q(B)$$

$$\neq S_q(A) + S_q(B) \quad (if\ \ q \neq 1)$$

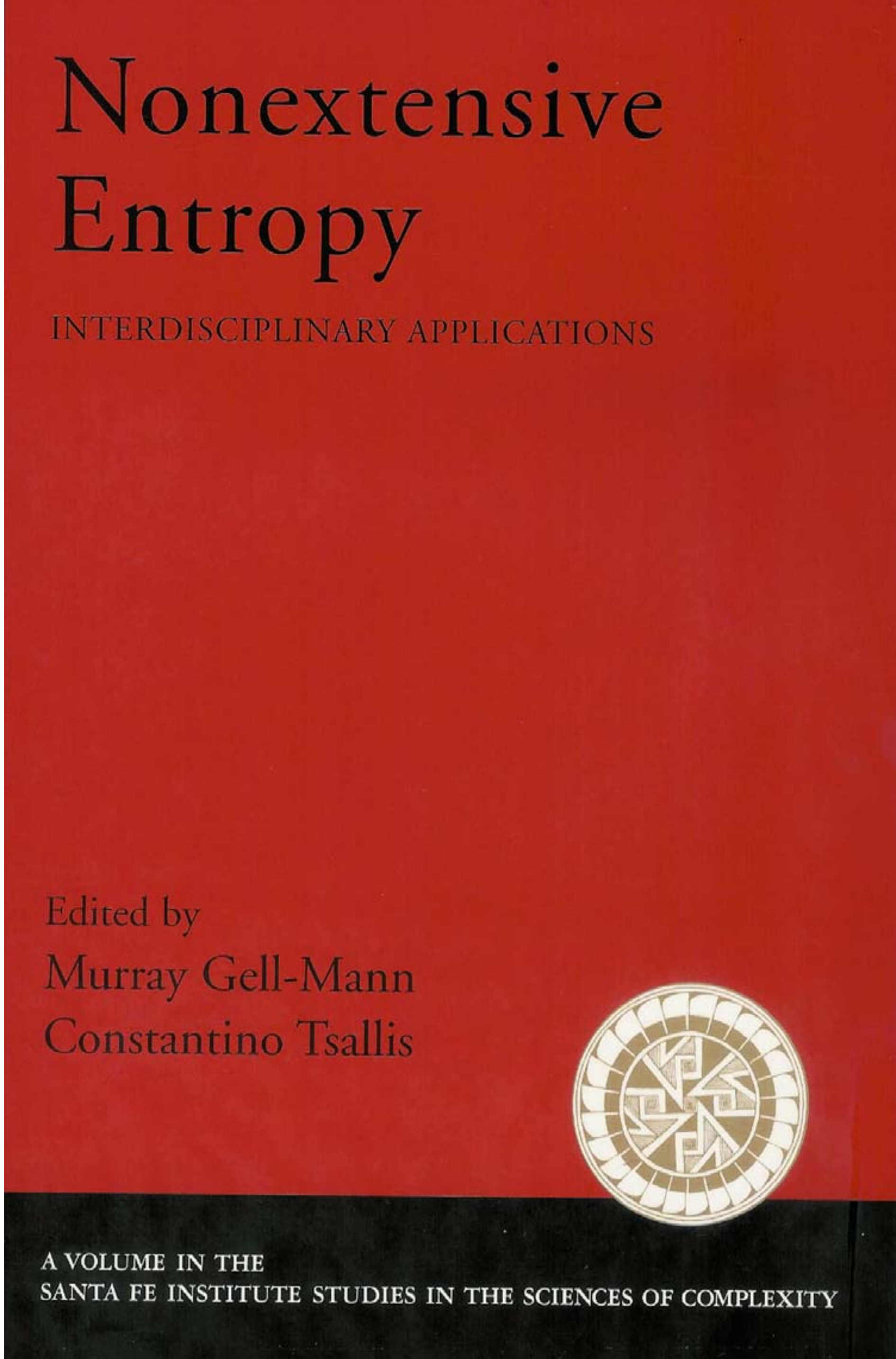
But if A and B are globally correlated,

then

$$S_q(A+B) = S_q(A) + S_q(B)$$

whereas

$$S_{BG}(A+B) \neq S_{BG}(A) + S_{BG}(B)$$



SPIN ½ XY FERROMAGNET WITH TRANSVERSE MAGNETIC FIELD:

$$\hat{\mathcal{H}} = -\sum_{j=1}^{N-1} \left[(1+\gamma)\hat{\sigma}_j^x \hat{\sigma}_{j+1}^x + (1-\gamma)\hat{\sigma}_j^y \hat{\sigma}_{j+1}^y + 2\lambda \hat{\sigma}_j^z \right]$$

$|\gamma| = 1 \quad \rightarrow \quad$ *Ising ferromagnet*

$0 < |\gamma| < 1 \quad \rightarrow \quad$ *anisotropic XY ferromagnet*

$\gamma = 0 \quad \rightarrow \quad$ *isotropic XY ferromagnet*

$\lambda \equiv$ *transverse magnetic field*

$L \equiv$ *length of a block within a* $N \rightarrow \infty$ *chain*

F. Caruso and C. T., cond-mat/0612032

$\rho_{\mathrm{N}} \equiv$ ground state $(T=0)$ of the N-system

$\Rightarrow \rho_N^2 = \rho_N \Rightarrow Tr\,\rho_N^2 = 1$

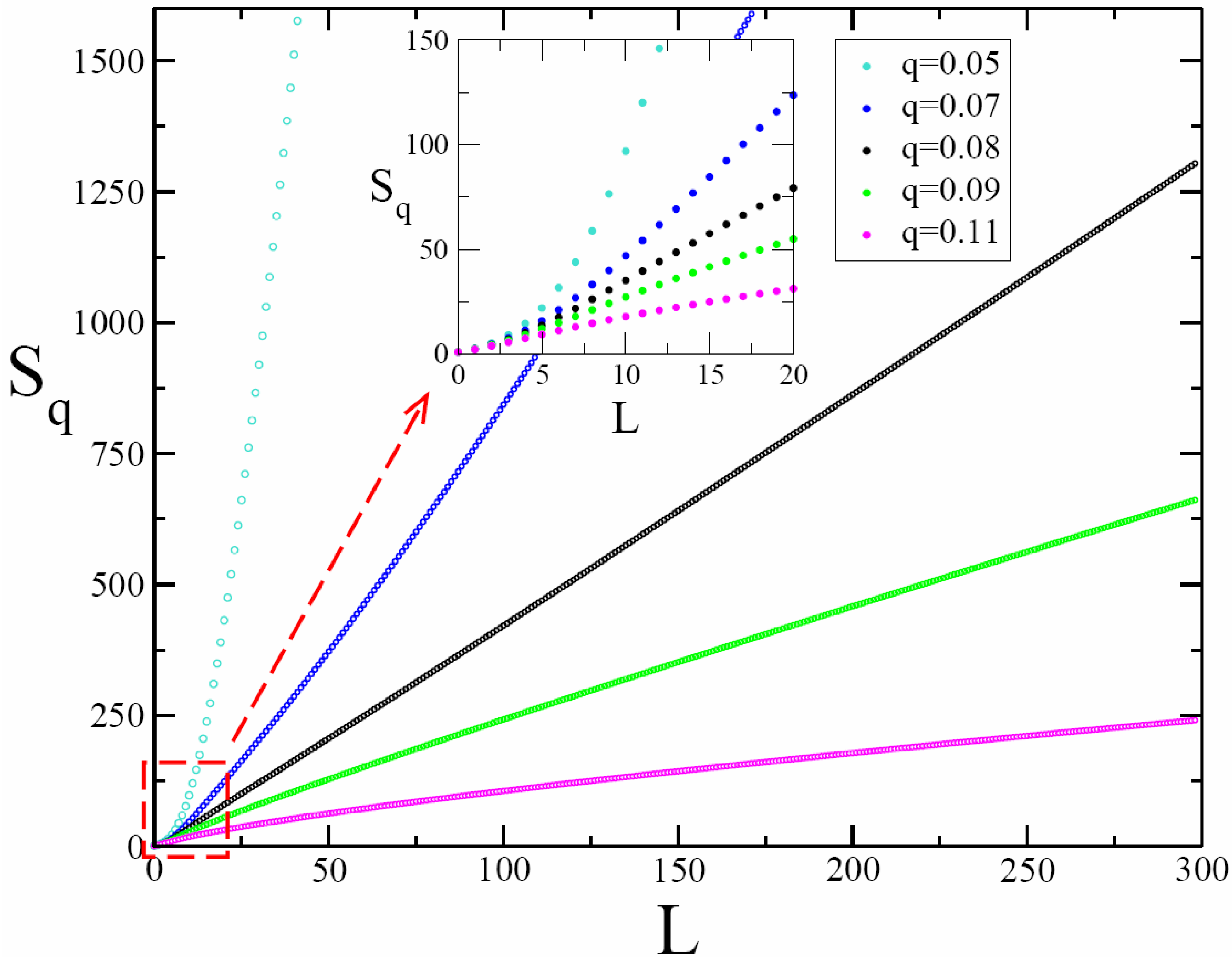
$\Rightarrow \rho_N$ is a pure state

$\Rightarrow S_q(N) = 0 \quad (\forall q,\ \forall N)$

Whereas $\rho_L \equiv Tr_{N-L}\ \rho_N$ satisfies $Tr\,\rho_L^2 < 1$

$\Rightarrow \rho_L$ is a mixed state

$\Rightarrow S_q(N,L) > 0$



F. Caruso and C. T., cond-mat/0612032

Using a Quantum Field Theory result

in P. Calabrese and J. Cardy, JSTAT P06002 (2004)

we obtain, at the critical transverse magnetic field,

$$q_{ent} = \frac{\sqrt{9+c^2} - 3}{c}$$

with $c \equiv$ *central charge in conformal field theory*

Hence

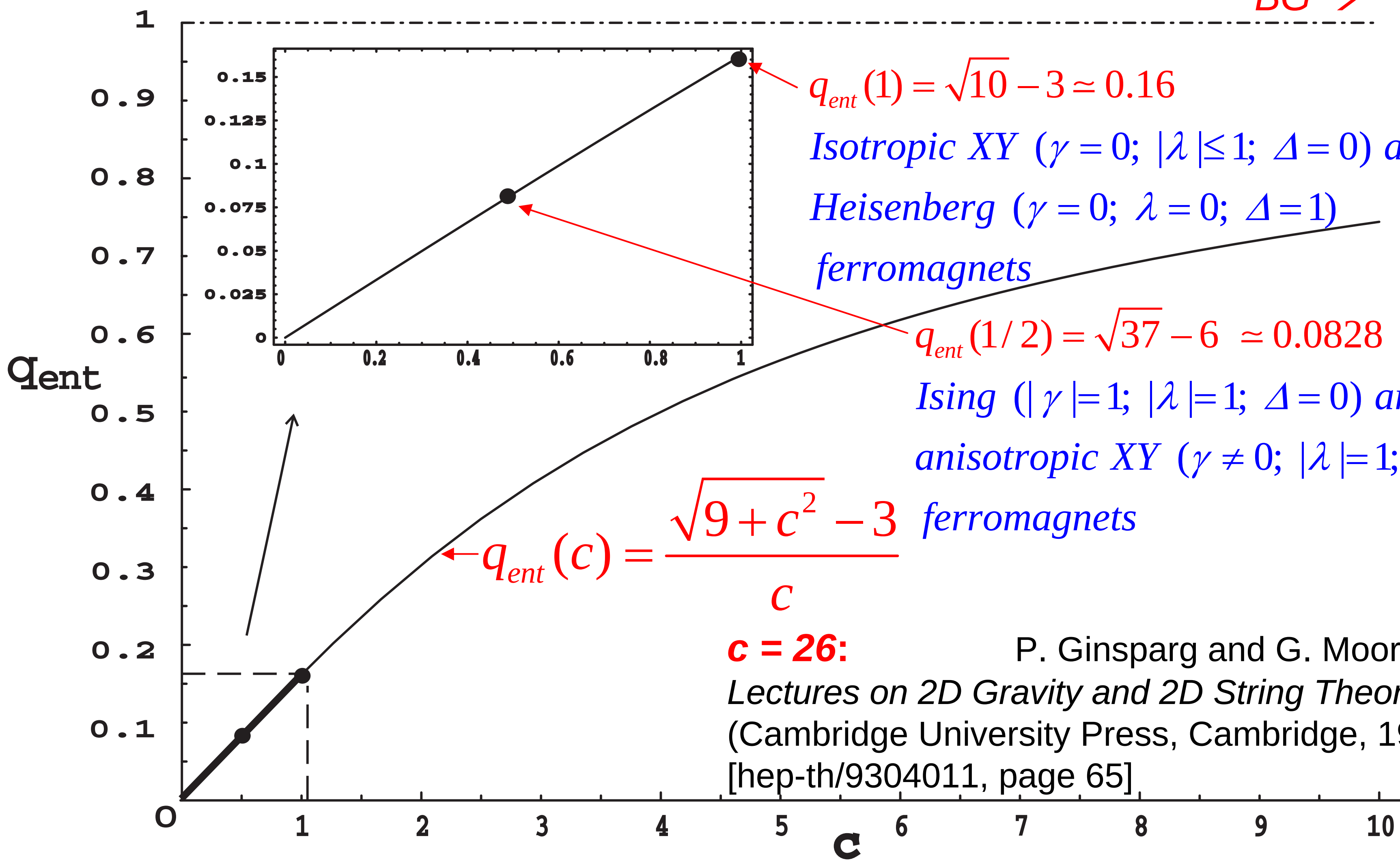
Ising and anisotropic XY ferromagnets $\Rightarrow \; c = \frac{1}{2} \;\Rightarrow\; q_{ent} = \sqrt{37} - 6 \;\approx 0.0828$

and

Isotropic XY ferromagnet $\Rightarrow \; c = 1 \;\Rightarrow\; q_{ent} = \sqrt{10} - 3 \;\approx 0.1623$

F. Caruso and C. T., cond-mat/0612032

$$H = -\sum_{i=1}^{N-1}\left[(1+\gamma)\,\sigma_i^x\sigma_{i+1}^x + (1-\gamma)\,\sigma_i^y\sigma_{i+1}^y + \Delta\,\sigma_i^z\sigma_{i+1}^z + 2\lambda\,\sigma_i^z\right]$$



***c* = 26:** P. Ginsparg and G. Moore
Lectures on 2D Gravity and 2D String Theory
(Cambridge University Press, Cambridge, 1993)
[hep-th/9304011, page 65]

F. Caruso and C. T., cond-mat/0612032

In other words,

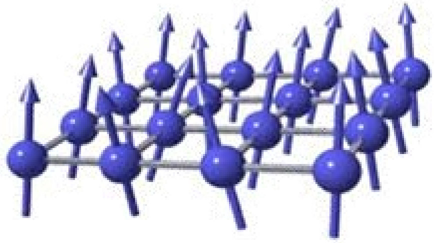
$$S_{\left[\sqrt{9+c^2}-3\right] c^{-1}}(L) \propto L \qquad \text{(extensive!)}$$

whereas

$$S_{BG}(L) \propto \ln L \qquad \text{(nonextensive!)}$$

- The entropic index q_{ent} characterizes universality classes (just like the central charge c does!)
- The slope $s_{q_{ent}}$ instead is not universal but depends on details
- The pair $(q_{ent},\ s_{q_{ent}})$ conveniently characterizes the nature of the quantum entanglement of the system

2-D quantum systems at T=0



Bosonic two-dimensional system of infinite coupled harmonic oscillators at T=0

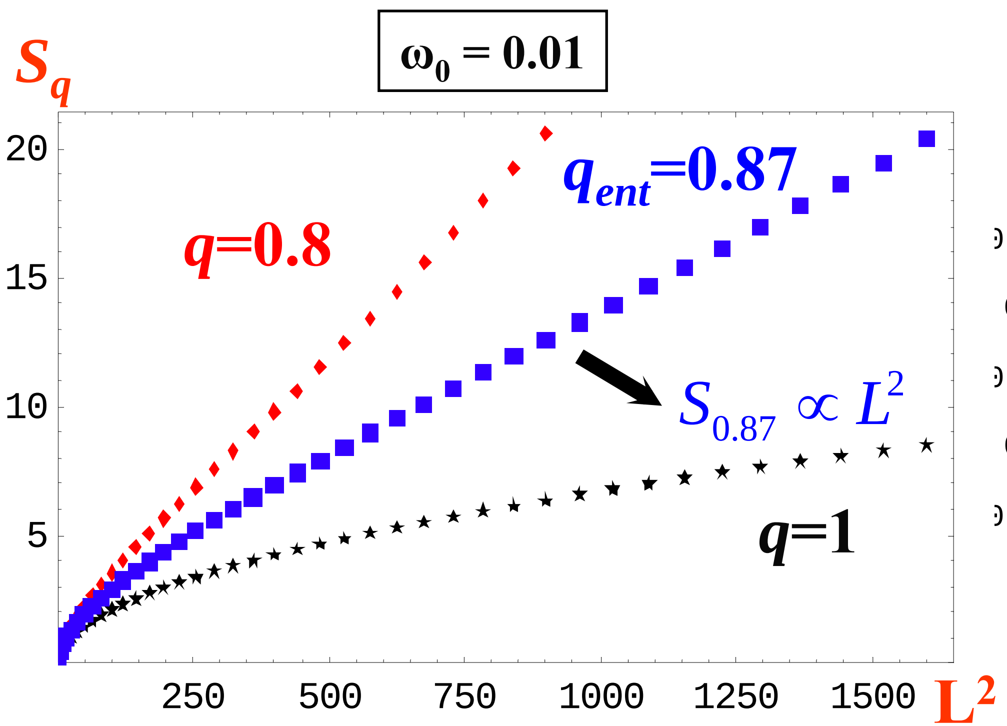
self-frequency

coordinate

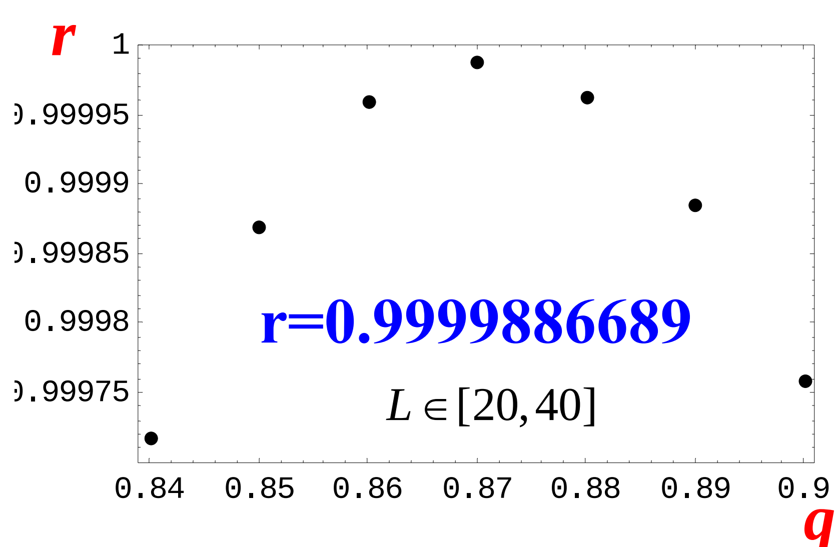
$$H = \frac{1}{2}\sum_{x,y} [\Pi^2_{x,y} + \omega_0^2\Phi^2_{x,y} + (\Phi_{x,y} - \Phi_{x+1,y})^2 + (\Phi_{x,y} - \Phi_{x,y+1})^2]$$

momentum

(the masses and coupling strengths are set to unity)



Extensivity for q_{ent}=0.87

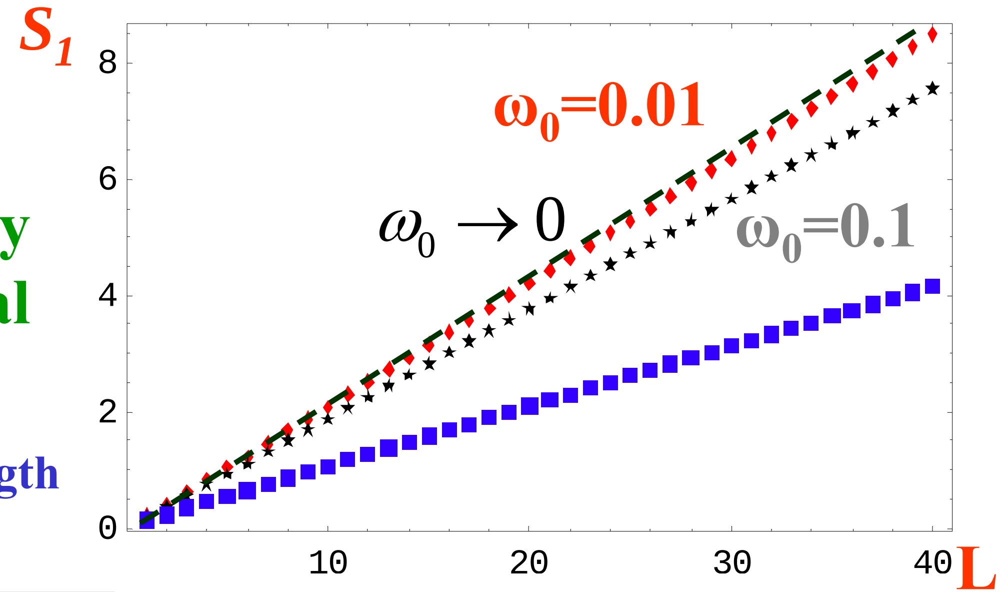


F. Caruso and C. T. (2007)

von Neumann entropy vs. entropy *Sq*

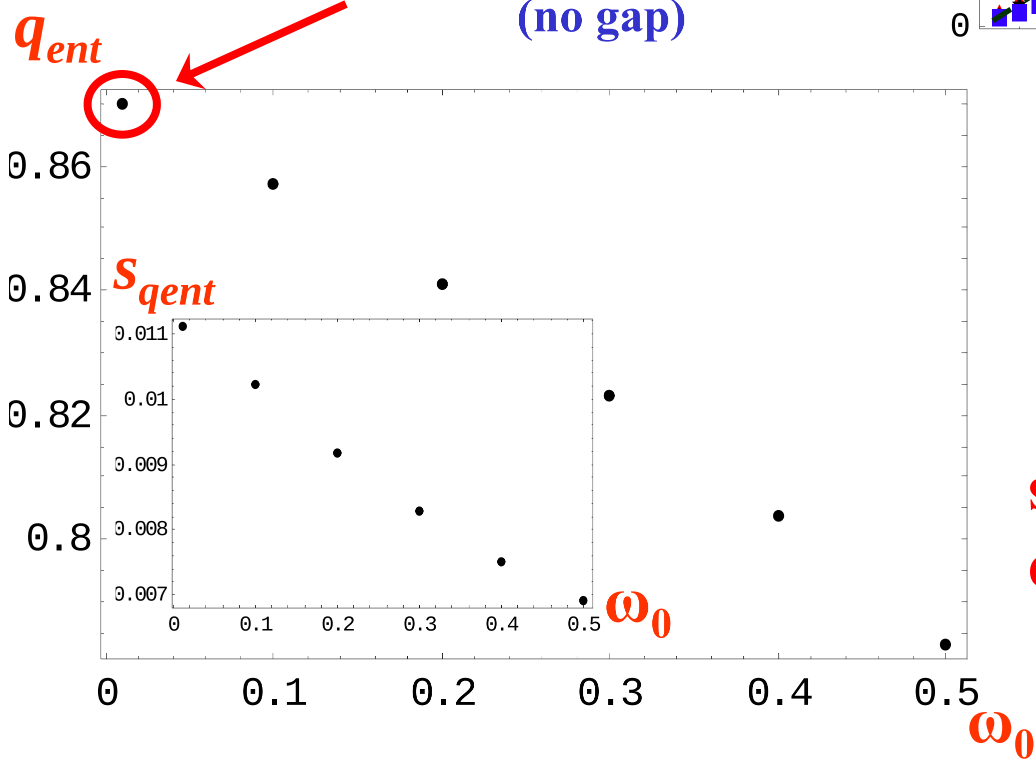
What happens for different values of the gap energy?

The von Neumann entropy violates thermodynamical extensivity



T. Barthel, M.-C. Chung, and U. Schollwöck, Phys. Rev. A **74**, 022329 (2006)

Max of q_{ent} for divergent correlation length (no gap)



Instead, the entropy *Sq* satisfies thermodynamical extensivity

Summarizing, for a wide class of quantum problems,

$$S_{BG}(L) \propto \ln L \qquad \text{for } d=1 \text{ quantum chains}$$

$$\propto L^{d-1} \qquad \text{for } d\text{-dimensional bosonic systems } (d>1)$$

[$d=3$ yields the famous black hole entropy]

$$\propto \frac{L^{d-1}-1}{d-1} \equiv \ln_{2-d} L \qquad \text{(conjecture for } d \geq 1\text{)} \qquad \text{(NONEXTENSIVE!)}$$

whereas, for the same class of quantum problems, we verify

$$S_{q_{ent}}(L) \propto L^{d} \quad (d \geq 1) \qquad \text{(EXTENSIVE!)}$$

- The entropic index q_{ent} characterizes universality classes

(just like the central charge c does!)

- The slope $s_{q_{ent}}$ instead is not universal but depends on details

- The pair $(q_{ent},\ s_{q_{ent}})$ conveniently characterizes the nature

of the quantum entanglement of the system

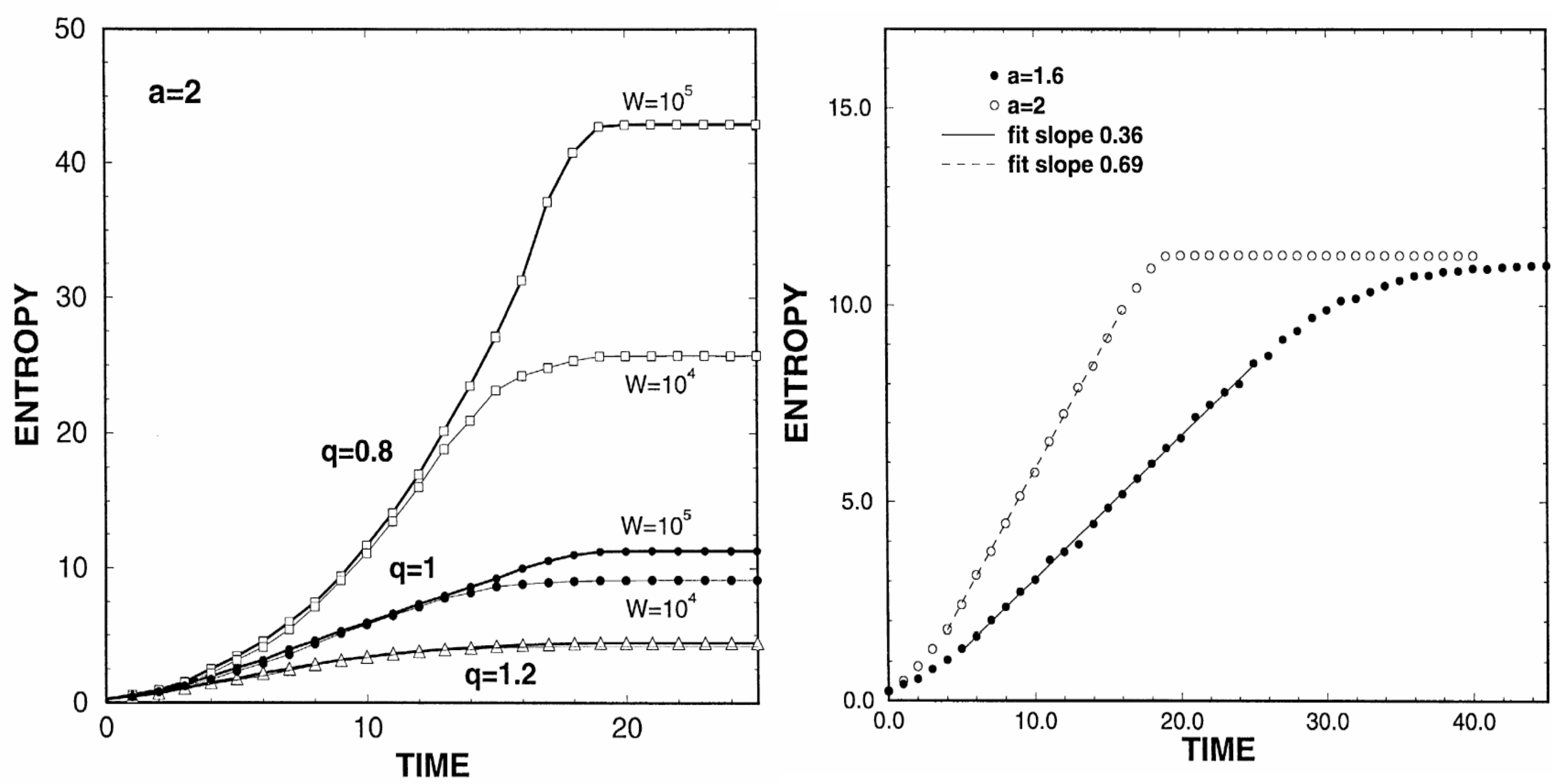
F. Caruso and C. T. (2007)

$$S_q(N,t) \; versus \;\; t$$

LOGISTIC MAP:

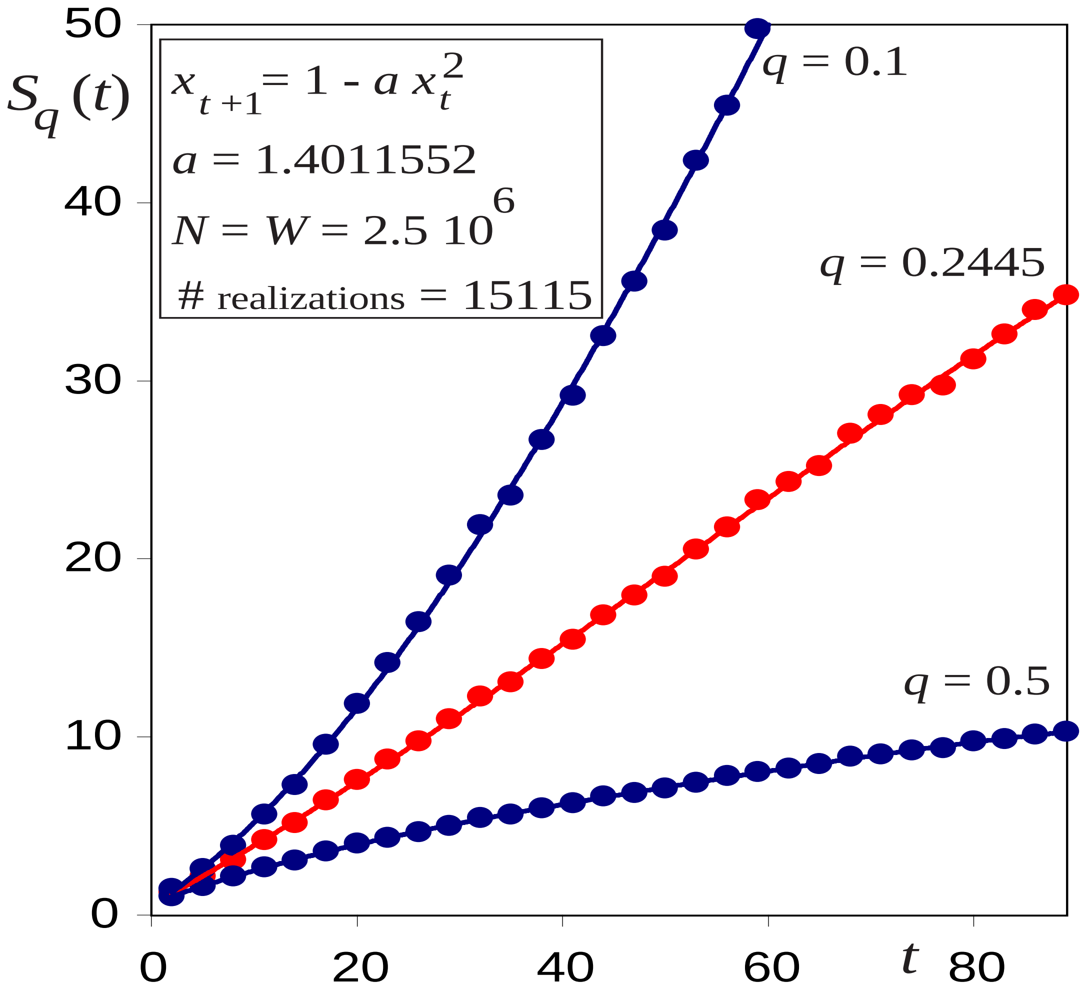
$$x_{t+1} = 1 - a\,x_t^2 \quad (0 \le a \le 2;\; -1 \le x_t \le 1;\; t = 0,1,2,\ldots)$$

(strong chaos, i.e., **positive** Lyapunov exponent)



V. Latora, M. Baranger, A. Rapisarda and C. T., Phys. Lett. A **273**, 97 (2000)

(weak chaos, i.e., **zero** Lyapunov exponent)



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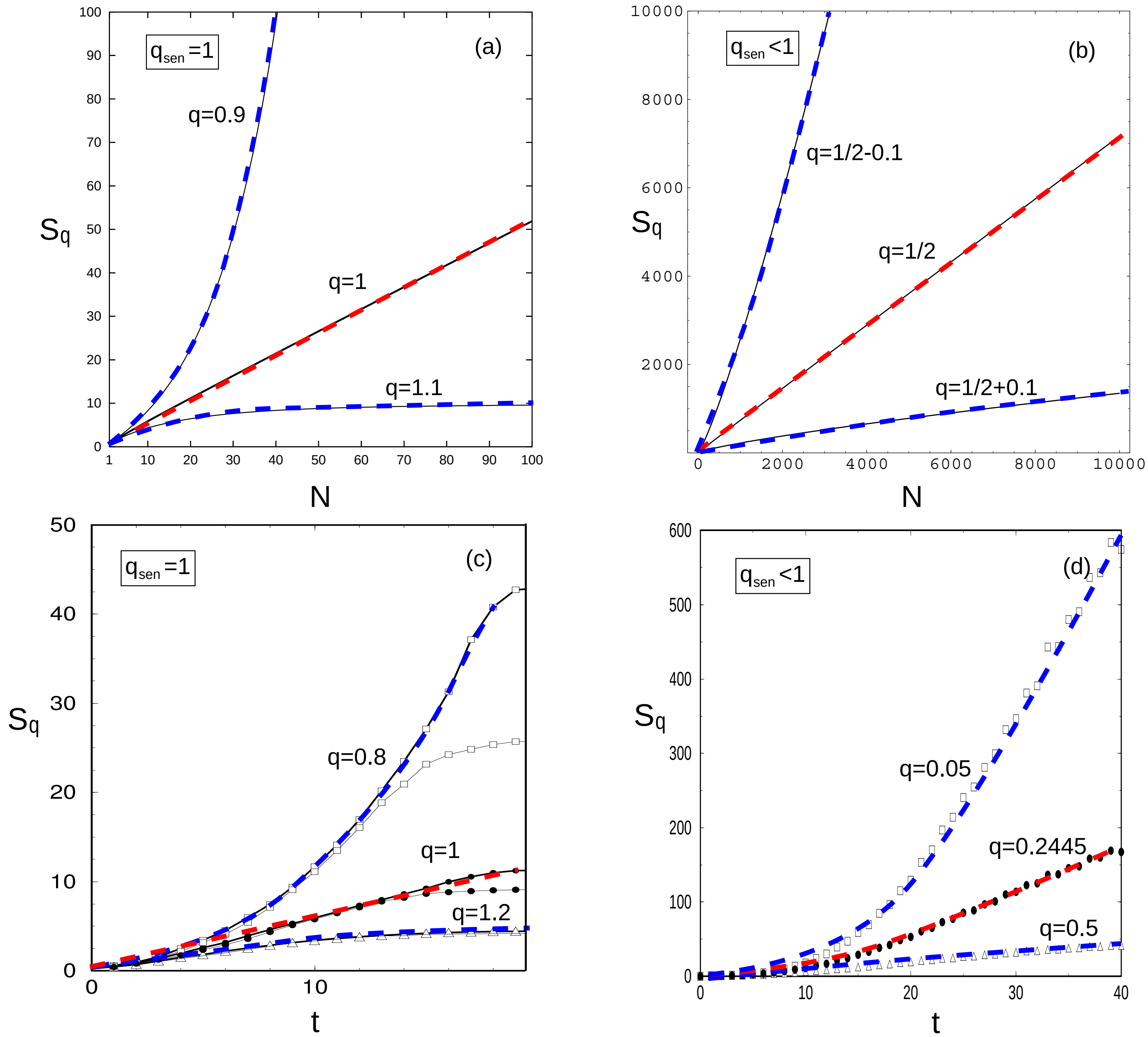
E.P. Borges, C. T. , G.F.J. Ananos and P.M.C. Oliveira, Phys. Rev. Lett. **89**, 254103 (2002)
F. Baldovin and A. Robledo, Phys. Rev. E **66**, R045104 (2002) and **69**, R045202 (2004)
G.F.J. Ananos and C. T. , Phys. Rev. Lett. **93**, 020601 (2004)
E. Mayoral and A. Robledo, Phys. Rev. E **72**, 026209 (2005), and references therein



C.T., M. Gell-Mann and Y. Sato
Europhysics News **36**, 186 (2005)

$q-$GENERALIZATION OF THE CENTRAL LIMIT THEOREM

M. Bologna, C. T. and P. Grigolini, Phys. Rev. E 62, 2213 (2000)

C. T., Milan J. Math. 73, 145 (2005)

C. T., Physica A 365, 7 (2006)

S. Umarov, C. T., M. Gell-Mann and S. Steinberg, cond-mat/0603593, 0606038, 0606040 and 0703533 (2006, 2007)

L.G. Moyano, C. T. and M. Gell-Mann, Europhys Lett 73, 813 (2006)

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H.J. Hilhorst and G. Schehr, J. Stat. Mech. (2007) P06003

A. Pluchino, A. Rapisarda and C. T., Europhys Lett 80, 26002 (2007)

C. Vignat and A. Plastino, J. Phys. A 40, F969 (2007)

W. Thistleton, J.A. Marsh, K. Nelson and C. T., IEEE 53 (12) (2007) in press

C. T. and S.M.D. Queiros, American Inst Phys Conf Proc 965 (2007) in press

ONE OF MANY CONNECTIONS OF THE CENTRAL LIMIT THEOREM WITH BOLTZMANN-GIBBS STATISTICAL MECHANICS

Optimization of

$$S = -\, k \int dx \; p(x) \; \ln[p(x)]$$

with

$$\int dx \; p(x) = 1$$

and

$$\langle E(x) \rangle \equiv \int dx \; p(x) \; E(x) = constant$$

yields

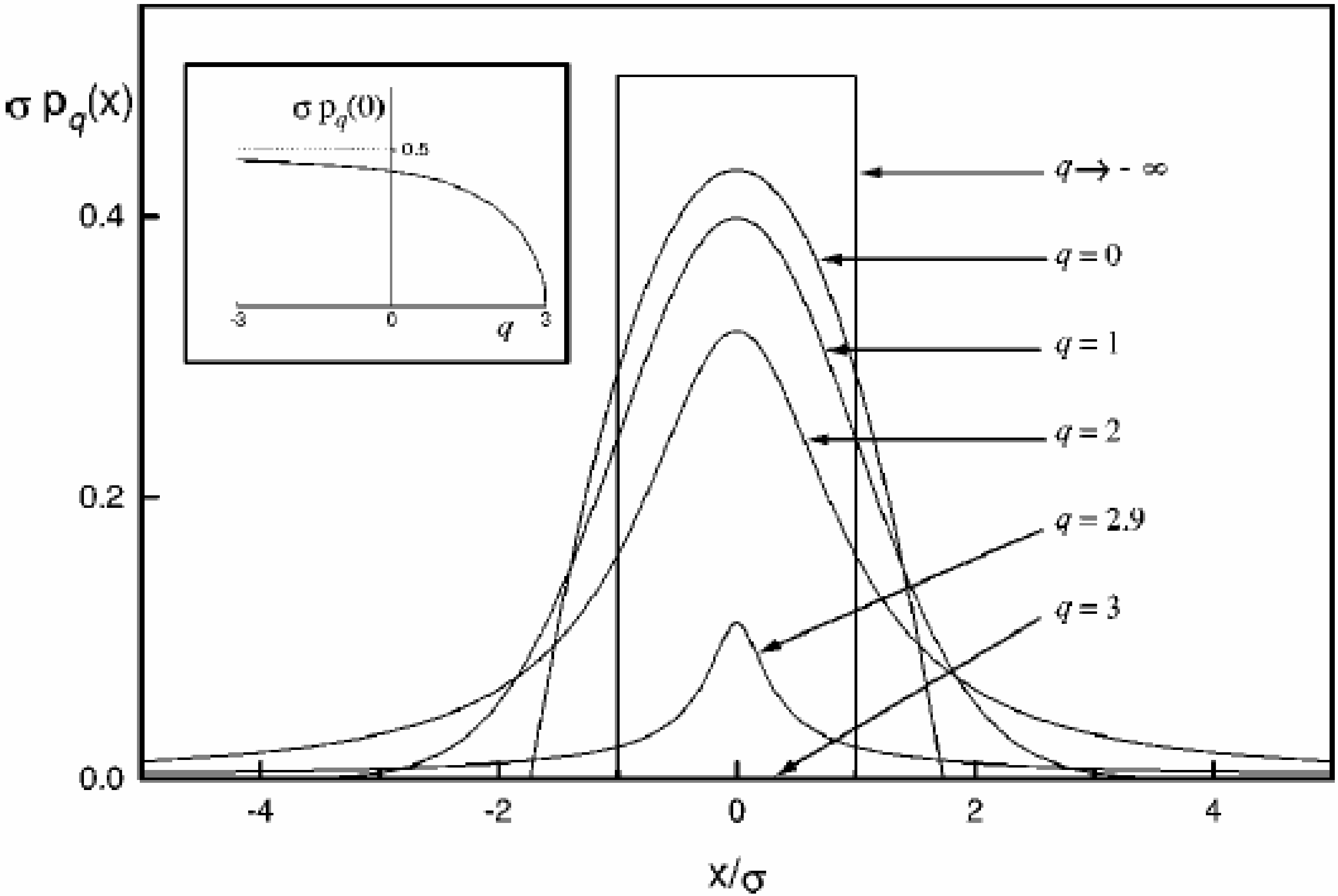
$$p(x) = \frac{e^{-\beta E(x)}}{\int dy \; e^{-\beta E(y)}}$$

(Boltzmann-Gibbs distribution for thermal equilibrium)

Example: $\langle x \rangle = 0$ *and* $\langle x^2 \rangle = constant$ *yields*

$$p(x) = \frac{e^{-\beta x^2}}{\int dy \; e^{-\beta y^2}} \qquad \text{(Gaussian distribution)}$$

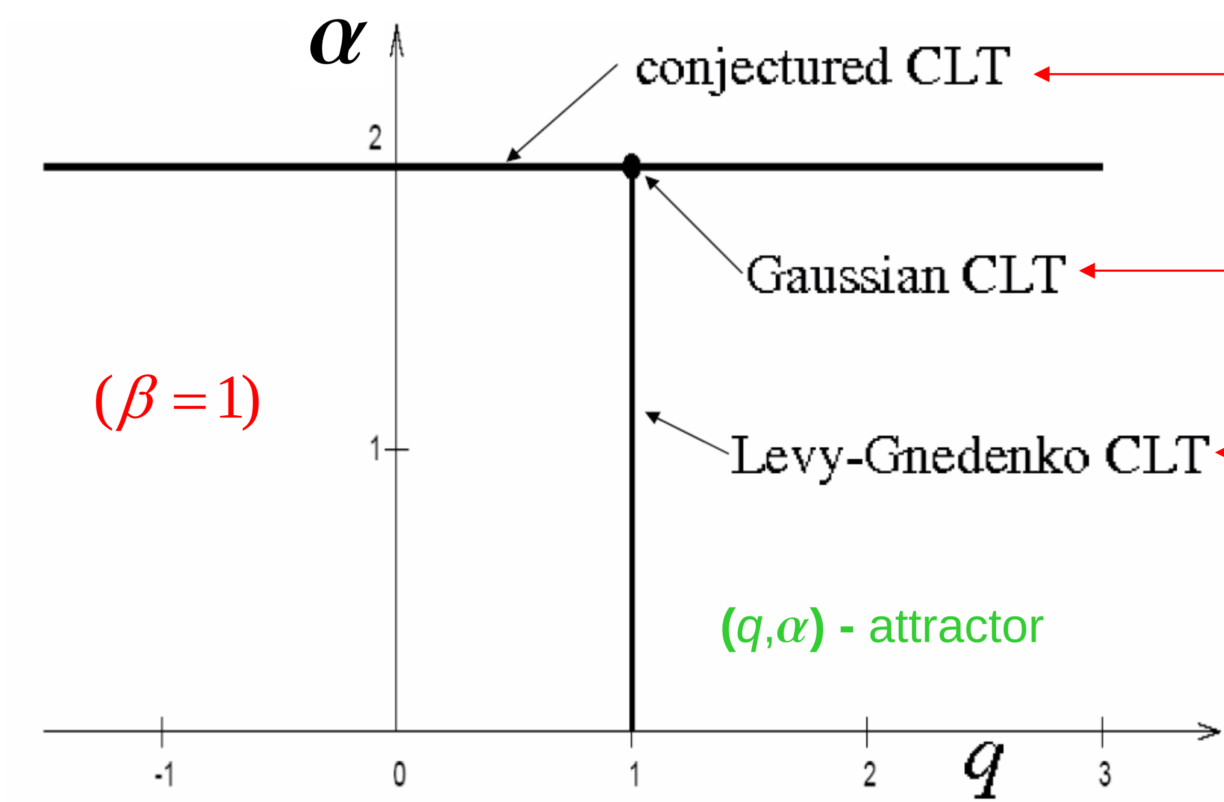
q-GAUSSIANS: $p_q(x) \propto e_q^{-(x/\sigma)^2} \equiv \dfrac{1}{\left[1+(q-1)\,(x/\sigma)^2\right]^{\frac{1}{q-1}}} \qquad (q<3)$



D. Prato and C. T., Phys Rev E **60**, 2398 (1999)

LOOKING FOR A *q* -GENERALIZED CENTRAL LIMIT THEOREM:

$$\frac{\partial^{\beta} p(x,t)}{\partial\, t^{\beta}} = D\, \frac{\partial^{\alpha}[p(x,t)]^{2-q}}{\partial\, |x|^{\alpha}} \qquad (0<\alpha\le 2;\; q<3;\; t\ge 0)$$



globally correlated variables;
finite q-variance;
q -Gaussian attractor

independent variables;
finite variance;
Gaussian attractor

independent variables;
divergent variance;
Levy attractor

M. Bologna, C. T. and P. Grigolini, Phys. Rev. E **62**, 2213 (2000)
C. T., Milan J. Math. **73**, 145 (2005)

q – PRODUCT:

L. Nivanen, A. Le Mehaute and Q.A. Wang, Rep. Math. Phys. **52**, 437 (2003)
E.P. Borges, Physica A **340**, 95 (2004)

The ***q* - product** is defined as follows:

$$x \otimes_q y \equiv \left[x^{1-q} + y^{1-q} - 1 \right]^{\frac{1}{1-q}}$$

$$Properties:$$

$$i) \quad x \otimes_1 y = x\, y$$

$$ii) \quad \ln_q (x \otimes_q y) = \ln_q x + \ln_q y$$

$$[whereas \;\; \ln_q (x\, y) = \ln_q x + \ln_q y + (1-q)(\ln_q x)(\ln_q y)]$$

q - GENERALIZED CENTRAL LIMIT THEOREM:

S. Umarov, C.T. and S. Steinberg [cond-mat/0603593]

q-Fourier transform:

$$F_q[f](\xi) \equiv \int_{-\infty}^{\infty} e_q^{ix\xi} \otimes_q f(x)\, dx = \int_{-\infty}^{\infty} e_q^{\frac{ix\xi}{[f(x)]^{1-q}}} f(x)\, dx$$

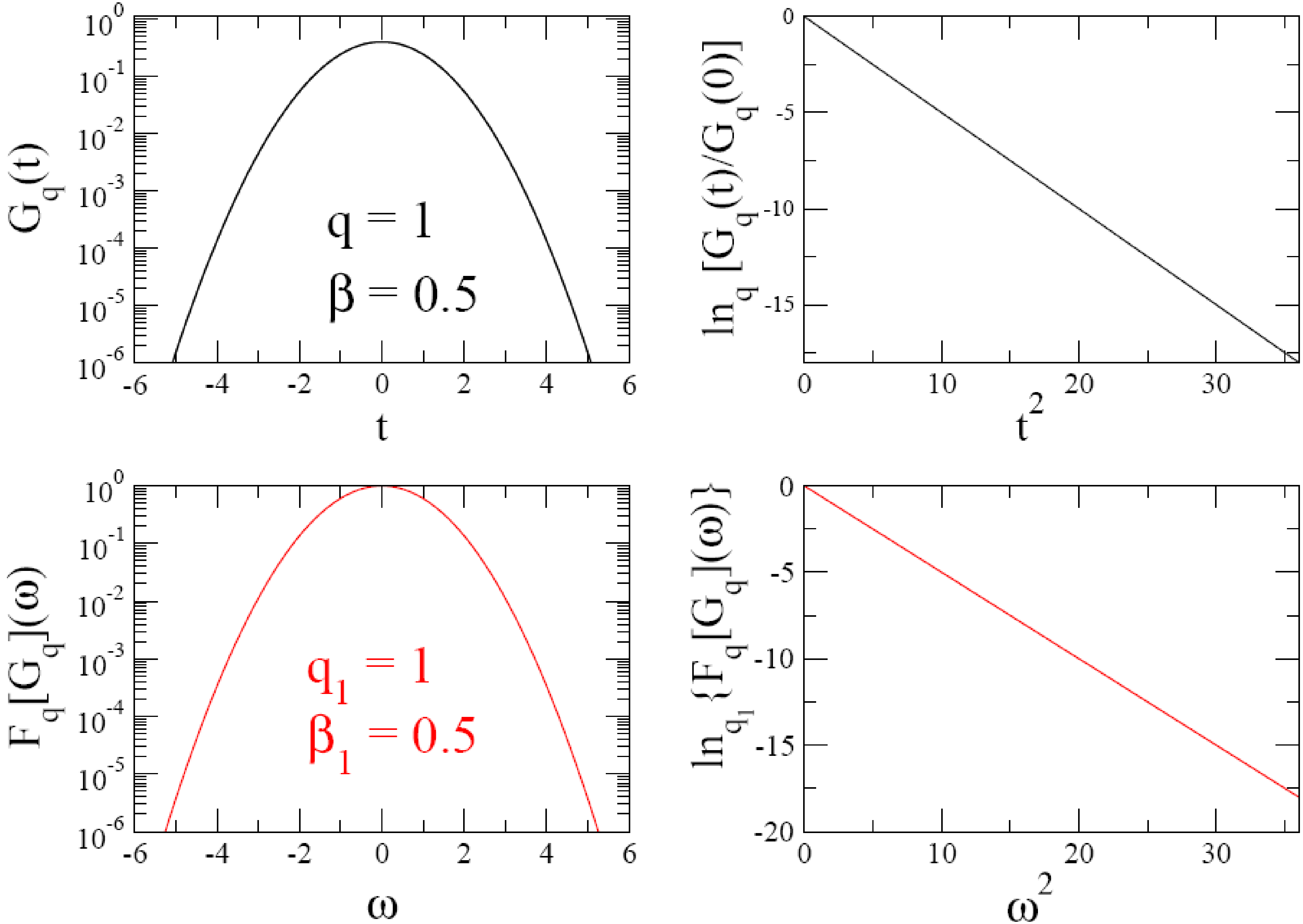
(nonlinear!)

$$q-FourierTransform\left[\frac{\sqrt{\beta}}{C_q}e_q^{-\beta\, t^2}\right]=e_{q_1}^{-\beta_1\,\omega^2}$$

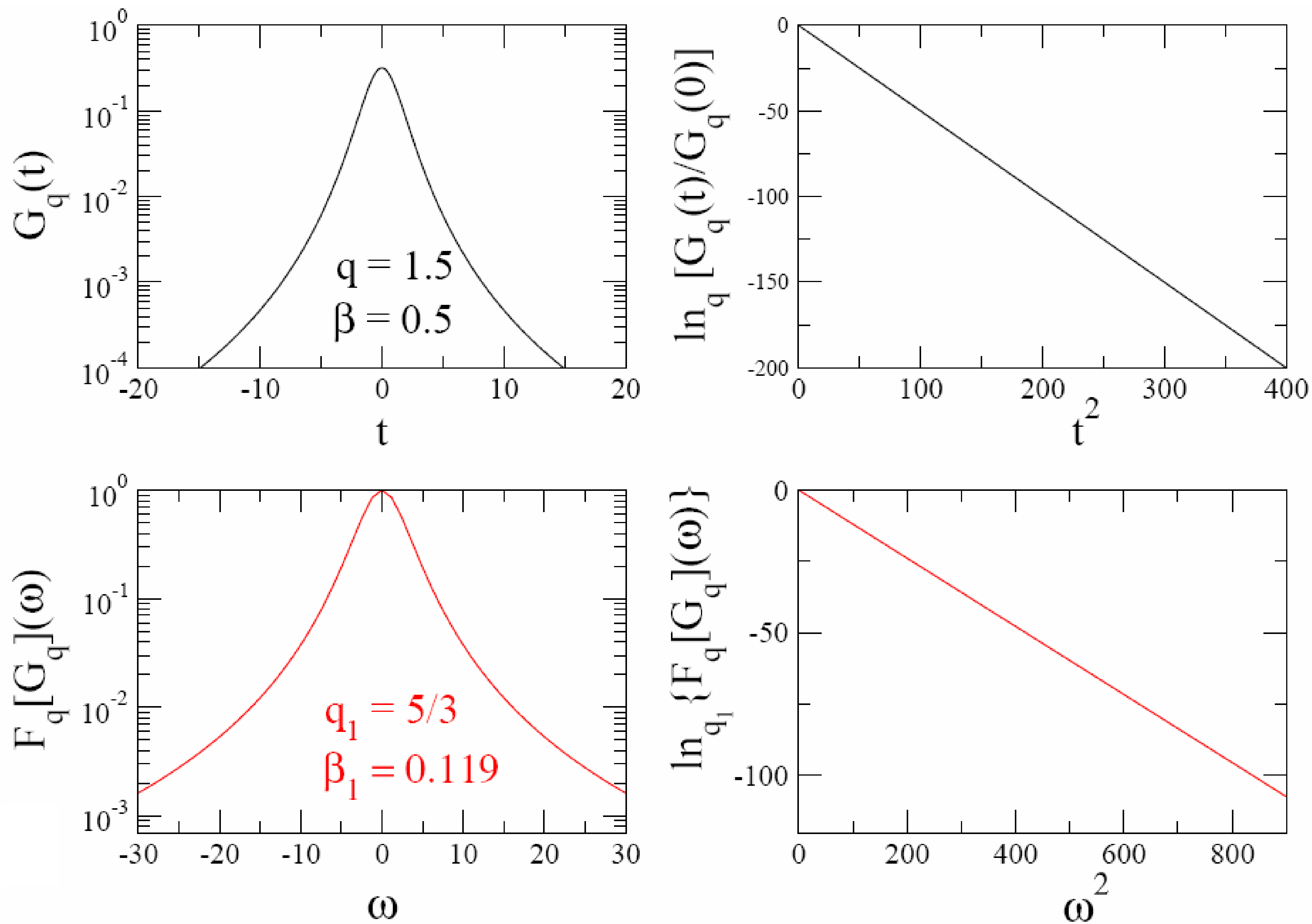
$$where \qquad q_1=\frac{1+q}{3-q}$$

$$and \qquad \beta_1=\frac{3-q}{8\beta^{2-q}C_q^{2(1-q)}} \quad\Leftrightarrow\quad \left(\beta_1\right)^{\frac{1}{\sqrt{2-q}}}\beta^{\sqrt{2-q}}=\left[\frac{3-q}{8C_q^{2(1-q)}}\right]^{\frac{1}{\sqrt{2-q}}}\equiv K(q)$$

$$with \qquad C_q=\begin{cases}\dfrac{2\sqrt{\pi}\,\Gamma\left(\dfrac{1}{q-1}\right)}{(3-q)\sqrt{(1-q)}\,\Gamma\left(\dfrac{3-q}{2(1-q)}\right)} & if \quad q<1\\ \sqrt{\pi} & if \quad q=1\\ \dfrac{\sqrt{\pi}\,\Gamma\left(\dfrac{3-q}{2(q-1)}\right)}{\sqrt{q-1}\,\Gamma\left(\dfrac{1}{q-1}\right)} & if \quad 1<q<3\end{cases}$$



$G_q(t)$
$q = 1$
$\beta = 0.5$
t
$\ln_q[G_q(t)/G_q(0)]$
t^2
$F_q[G_q](\omega)$
$q_1 = 1$
$\beta_1 = 0.5$
ω
$\ln_{q_1}\{F_q[G_q](\omega)\}$
ω^2

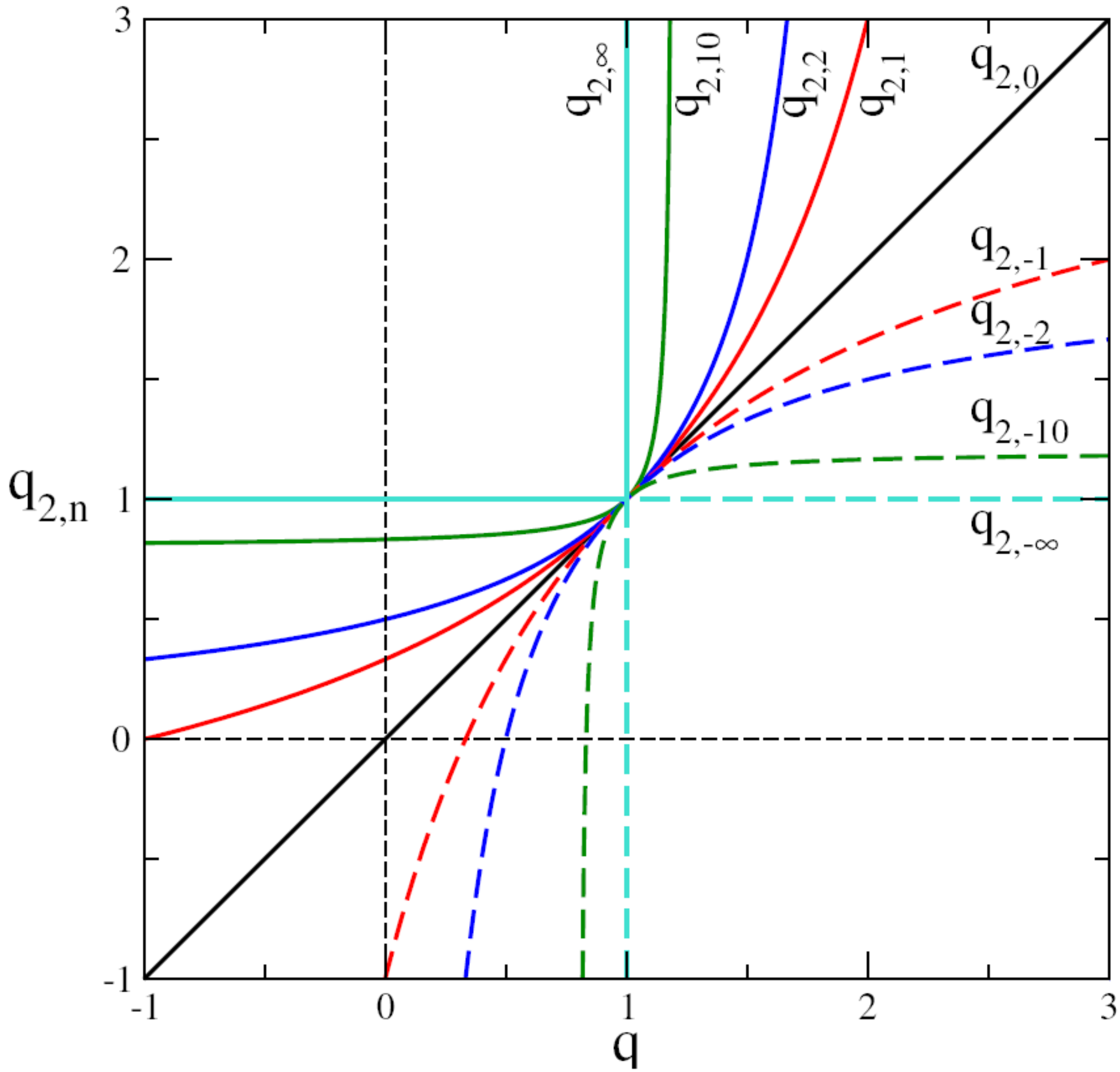


$G_q(t)$
t
q = 1.5
β = 0.5
$\ln_q[G_q(t)/G_q(0)]$
t^2
$F_q[G_q](\omega)$
ω
$q_1 = 5/3$
$\beta_1 = 0.119$
$\ln_{q_1}\{F_q[G_q](\omega)\}$
ω^2

ALGEBRA ASSOCIATED WITH *q* -GENERALIZED CENTRAL LIMIT THEOREMS:

$$\frac{\alpha}{1-q_{\alpha,n}} = \frac{\alpha}{1-q} + n$$

$$(n = 0, \ \pm 1, \ \pm 2, ...)$$



S. Umarov, C.T., M. Gell-Mann and S. Steinberg (2006), cond-mat/0606040

q - GENERALIZED CENTRAL LIMIT THEOREM:

S. Umarov, C.T. and S. Steinberg [cond-mat/0603593]

q-independence:

Two random variables X *[with density* $f_X(x)$ *] and* Y *[with density* $f_Y(y)$*]* *having zero* $q-$*mean values are said* q-*independent if*

$$F_q[\mathrm{X+Y}](\xi) = F_q[\mathrm{X}](\xi) \otimes_{\frac{1+q}{3-q}} F_q[\mathrm{Y}](\xi) \quad ,$$

i.e., if

$$\int_{-\infty}^{\infty} dz\; e_q^{iz\xi} \otimes_q f_{X+Y}(z) = \left[\int_{-\infty}^{\infty} dx\; e_q^{ix\xi} \otimes_q f_X(x)\right] \otimes_{(1+q)/(3-q)} \left[\int_{-\infty}^{\infty} dy\; e_q^{iy\xi} \otimes_q f_Y(y)\right],$$

with

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\; h(x,y)\; \delta(x+y-z) = \int_{-\infty}^{\infty} dx\; h(x,z-x) = \int_{-\infty}^{\infty} dy\; h(z-y,y)$$

where $h(x,y)$ *is the joint density.*

$$q\text{-}\mathit{independence\ means} \begin{cases} \mathit{independence} & \mathit{if}\;\; q=1\,,\;\; \mathit{i.e.,}\;\; h(x,y) = f_X(x)\, f_Y(y) \\ \mathit{global\ correlation} & \mathit{if}\;\; q \neq 1\,,\;\; \mathit{i.e.,}\;\; h(x,y) \neq f_X(x)\, f_Y(y) \end{cases}$$

A random variable X is said to have a (q,α)-stable distribution $L_{q,\alpha}(x)$ if its q-Fourier transform has the form $a\ e_{q_1}^{-b\,|\xi|^\alpha}$

$[a>0,\ b>0,\ 0<\alpha\le 2,\ q_1\equiv(q+1)/(3-q)]$

i.e., if

$$F_q[L_{q,\alpha}](\xi)\ \equiv\ \int_{-\infty}^{\infty} e_q^{ix\xi}\otimes_q L_{q,\alpha}(x)\,dx = \int_{-\infty}^{\infty} e_q^{\frac{ix\xi}{[L_{q,\alpha}(x)]^{1-q}}} L_{q,\alpha}(x)\,dx\ = a\ e_{q_1}^{-b\,|\xi|^\alpha}$$

$L_{1,2}(x)\equiv G(x)$ (Gaussian)

$L_{1,\alpha}(x)\equiv L_\alpha(x)$ (α − stable Levy distribution)

$L_{q,2}(x)\equiv G_q(x)$ (q − Gaussian)

S. Umarov, C. T., M. Gell-Mann and S. Steinberg (2006)

cond-mat/0606038 and cond-mat/0606040

CENTRAL LIMIT THEOREM

$N^{1/[\alpha(2-q)]}$ - *scaled attractor* $\mathbb{F}(x)$ *when summing* $N \to \infty$ q - *independent identical random variables*

with symmetric distribution $f(x)$ *with* $\sigma_Q \equiv \int dx\, x^2 [f(x)]^Q / \int dx\, [f(x)]^Q$ $\left(Q \equiv 2q-1,\ q_1 = \frac{1+q}{3-q} \right)$

	$q=1$ [*independent*]	$q \neq 1$ (*i.e.*, $Q \equiv 2q-1 \neq 1$) [*globally correlated*]
$\sigma_Q < \infty$ $(\alpha = 2)$	$\mathbb{F}(x) =$ *Gaussian* $G(x)$, *with same* σ_1 *of* $f(x)$ Classic CLT	$\mathbb{F}(x) = G_q(x) \equiv G_{(3q_1-1)/(1+q_1)}(x)$, *with same* σ_Q *of* $f(x)$ $G_q(x) \sim \begin{cases} G(x) & \text{if } \lvert x \rvert << x_c(q,2) \\ f(x) \sim C_q / \lvert x \rvert^{2/(q-1)} & \text{if } \lvert x \rvert >> x_c(q,2) \end{cases}$ *with* $\lim_{q \to 1} x_c(q,2) = \infty$ S. Umarov, C. T. and S. Steinberg (2006) [cond-mat/0603593]
$\sigma_Q \to \infty$ $(0 < \alpha < 2)$	$\mathbb{F}(x) =$ *Levy distribution* $L_\alpha(x)$, *with same* $\lvert x \rvert \to \infty$ *behavior* $L_\alpha(x) \sim \begin{cases} G(x) \\ \quad \text{if } \lvert x \rvert << x_c(1,\alpha) \\ f(x) \sim C_\alpha / \lvert x \rvert^{1+\alpha} \\ \quad \text{if } \lvert x \rvert >> x_c(1,\alpha) \end{cases}$ *with* $\lim_{\alpha \to 2} x_c(1,\alpha) = \infty$ Levy-Gnedenko CLT	$\mathbb{F}(x) = L_{q,\alpha}$, *with same* $\lvert x \rvert \to \infty$ *asymptotic behavior* $L_{q,\alpha} \sim \begin{cases} G_{\frac{2(1-q)-\alpha(1+q)}{2(1-q)-\alpha(3-q)},\, \alpha}(x) \sim C^*_{q,\alpha} / \lvert x \rvert^{\frac{2(1-q)-\alpha(3-q)}{2(1-q)}} \\ \quad (\textit{intermediate regime}) \\ G_{\frac{2\alpha q-\alpha+3}{\alpha+1},\, 2}(x) \sim C^L_{q,\alpha} / \lvert x \rvert^{(1+\alpha)/(1+\alpha q-\alpha)} \\ \quad (\textit{distant regime}) \end{cases}$ S. Umarov, C. T., M. Gell-Mann and S. Steinberg (2006) [cond-mat/0606038] and [cond-mat/0606040]

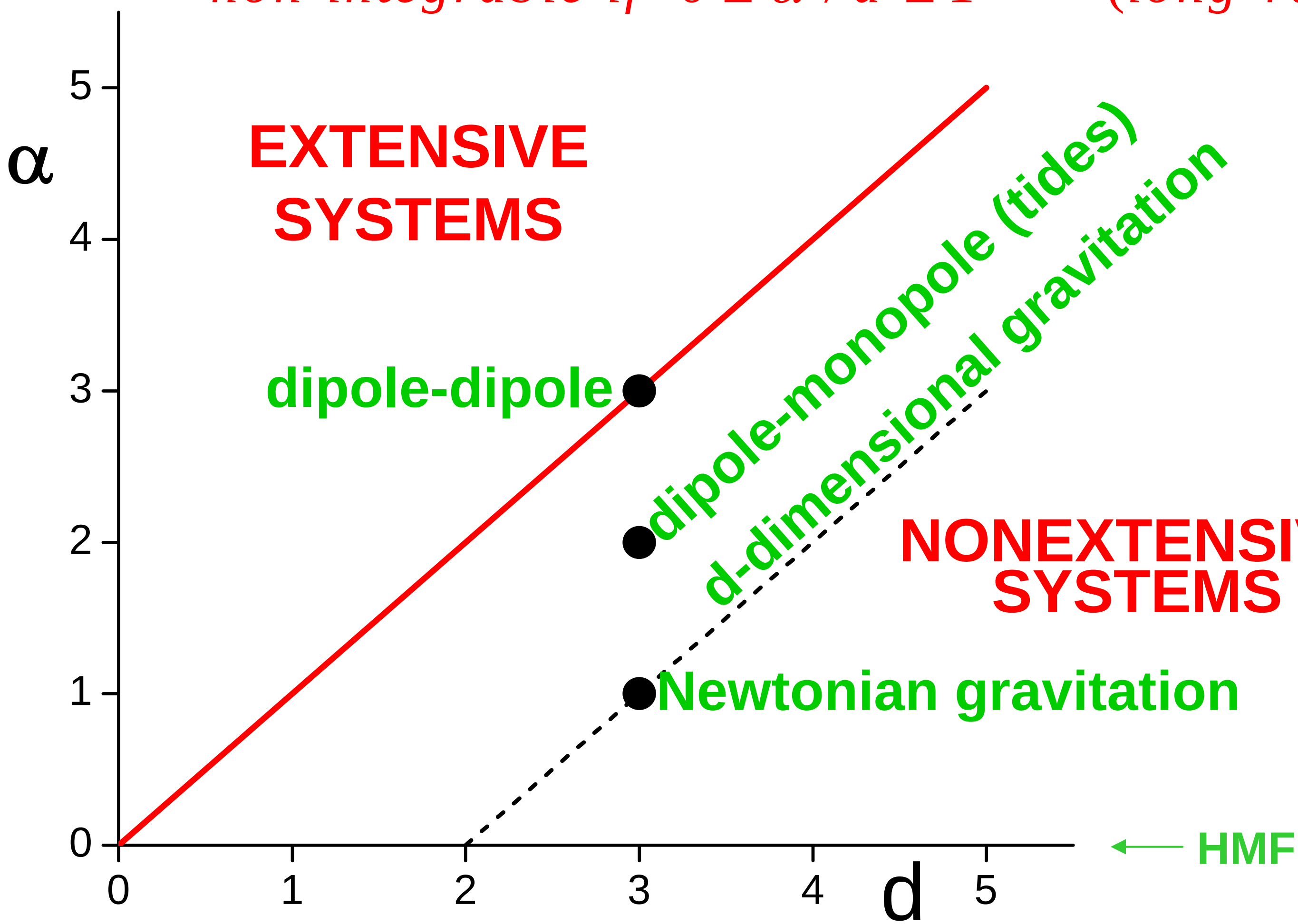
Connections with Hamiltonian and more complex systems

$$V(\vec{r}) \sim -\frac{A}{r^{\alpha}} \qquad (r \to \infty)$$

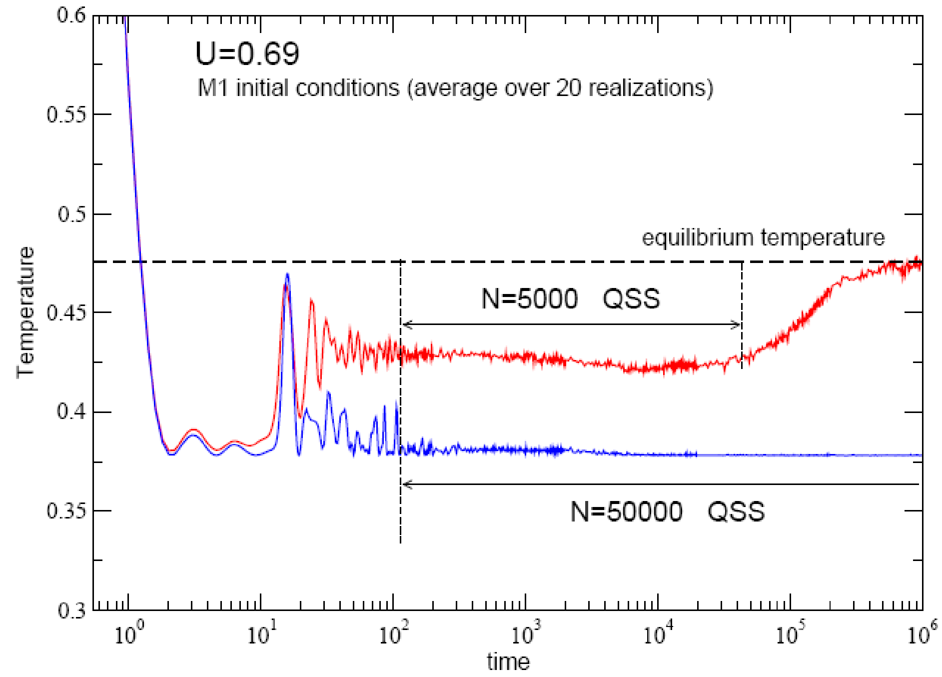
$$(A > 0, \quad \alpha \geq 0)$$

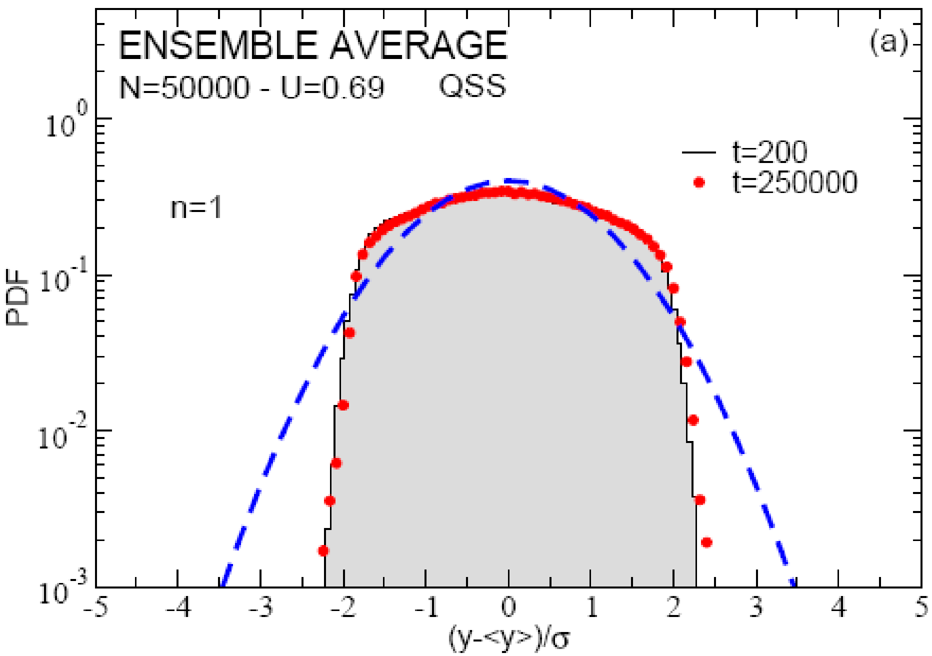
integrable if $\alpha / d > 1$ (*short-ranged*)

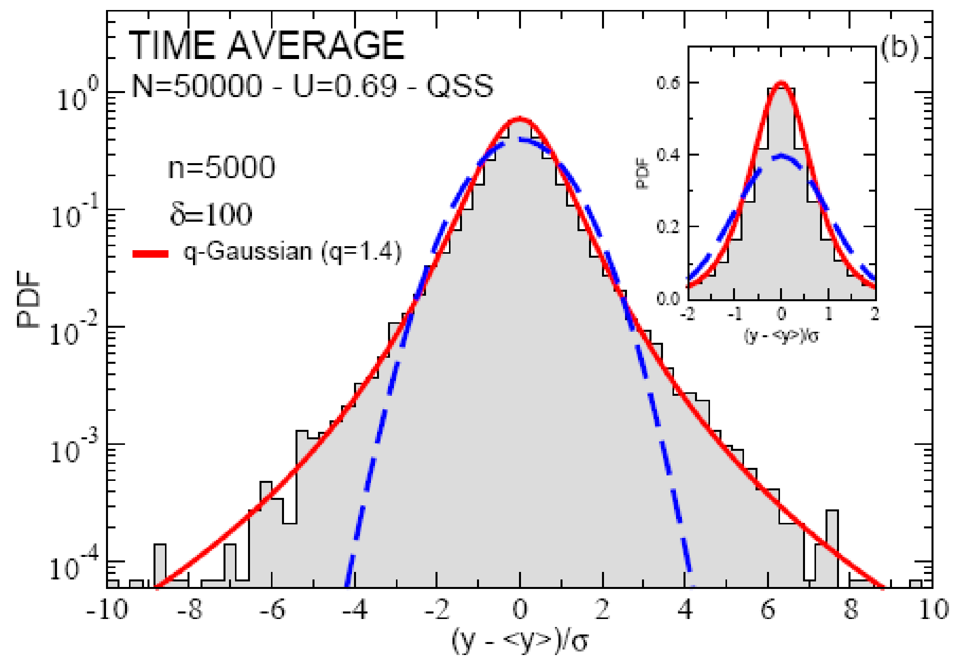
non-integrable if $0 \leq \alpha / d \leq 1$ (*long-ranged*)



HMF MODEL







A. Pluchino, A. Rapisarda and C. T., Europhys Lett **80**, 26002 (2007)

COLD ATOMS IN DISSIPATIVE OPTICAL LATTICES:

Theoretical predictions by E. Lutz, Phys Rev A **67**, 051402(R) (2003):

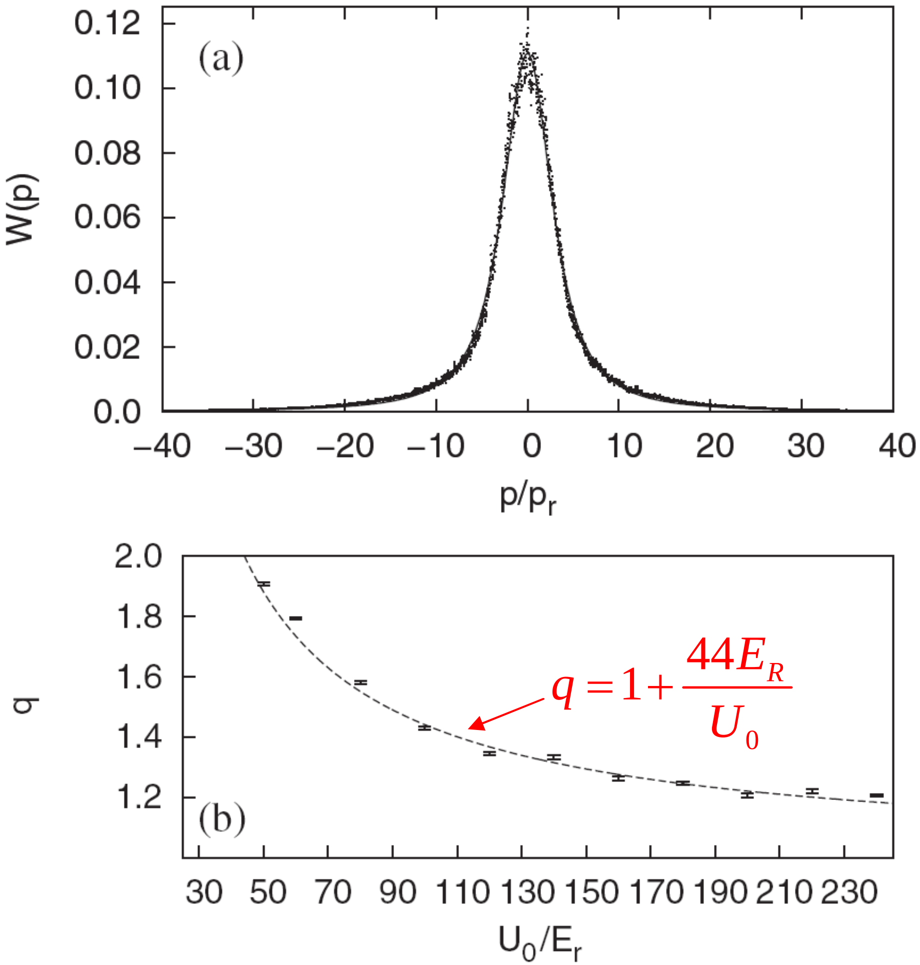
(i) The distribution of atomic velocities is a *q*-Gaussian;

(ii) $q = 1 + \frac{44 E_R}{U_0}$ where $E_R \equiv$ recoil energy

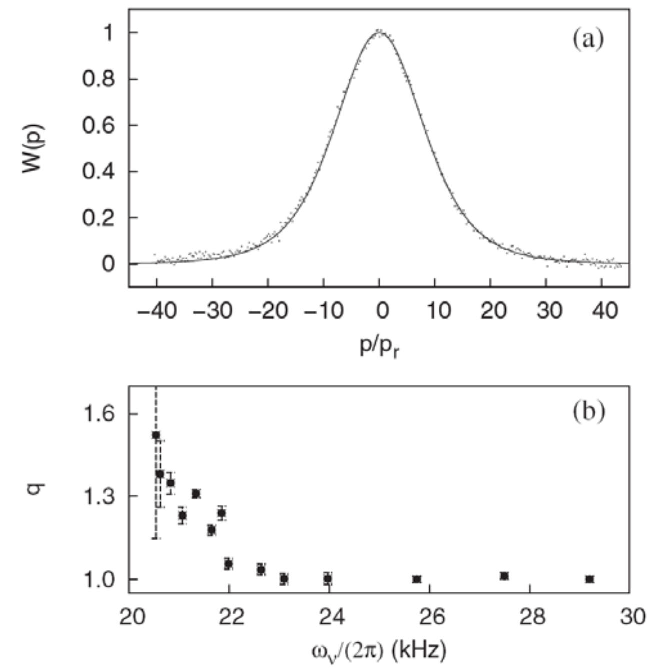
$U_0 \equiv$ potential depth

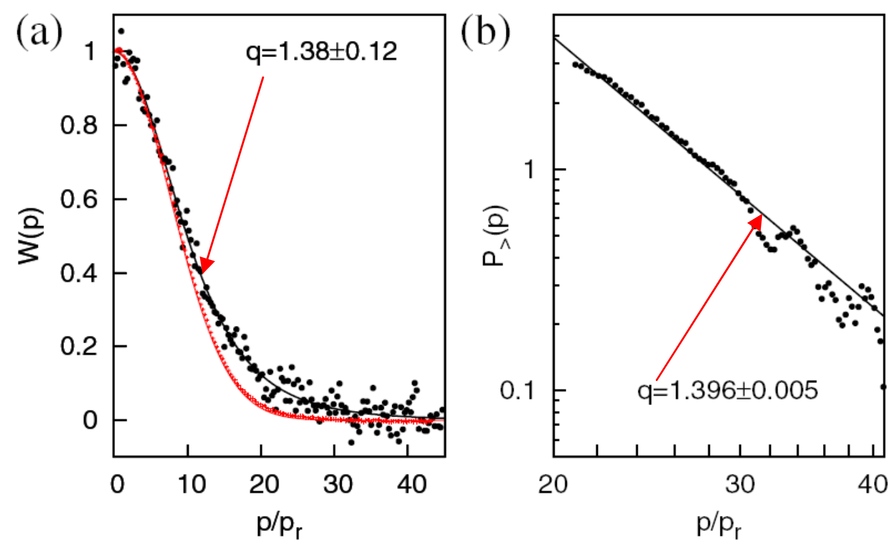
Experimental and computational verifications

by P. Douglas, S. Bergamini and F. Renzoni, Phys Rev Lett **96**, 110601 (2006)



(Computational verification: quantum Monte Carlo simulations)



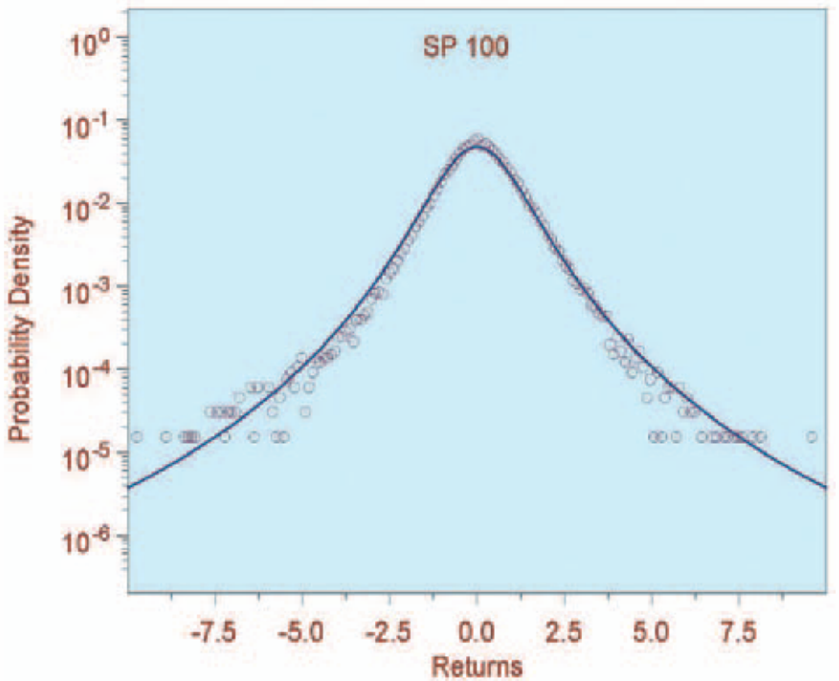


(Experimental verification)

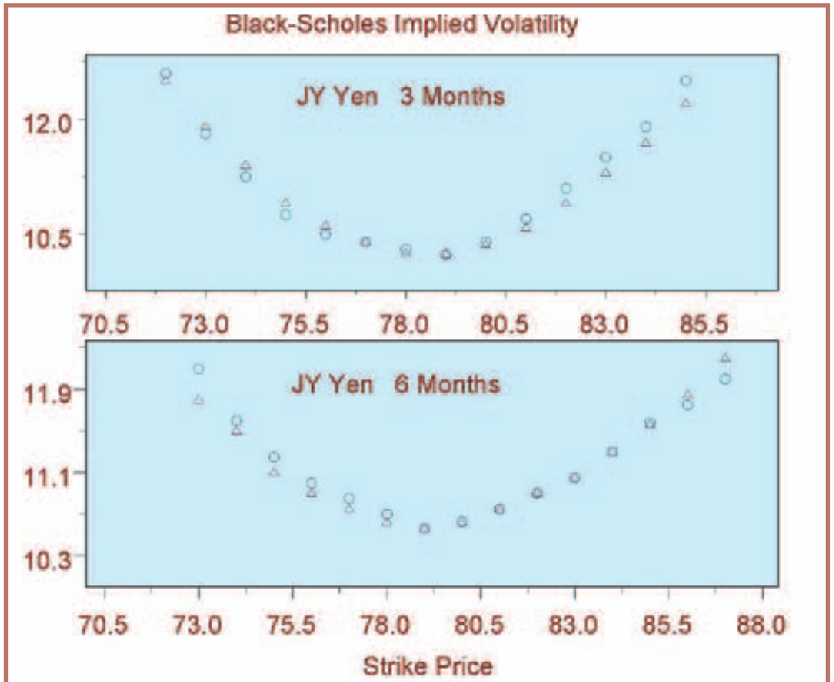
Connections with Economics

q-GENERALIZED BLACK-SCHOLES EQUATION:

L Borland, Phys Rev Lett **89**, 098701 (2002), and Quantitative Finance **2**, 415 (2002)
L Borland and J-P Bouchaud, Quantitative Finance **4**, 499 (2004)
L Borland, Europhys News **36**, 228 (2005)
See also H Sakaguchi, J Phys Soc Jpn **70**, 3247 (2001)
C Anteneodo and CT, J Math Phys **44**, 5194 (2003)



▲ Fig.2: The empirical distribution of daily returns from the stocks comprising the SP 100 (red) is fit very well by a q-Gaussian with $q = 1.4$ (blue).



▲ Fig.3: Theoretical implied Black-Scholes volatilities from the $q = 1.4$ model (triangles) match empirical ones (circles) very well, across all strikes and for different times to expiration.

[*REMARK: Student t − distributions are the particular case of q − Gaussians when* $q = \frac{n+3}{n+1}$ *with n integer*]

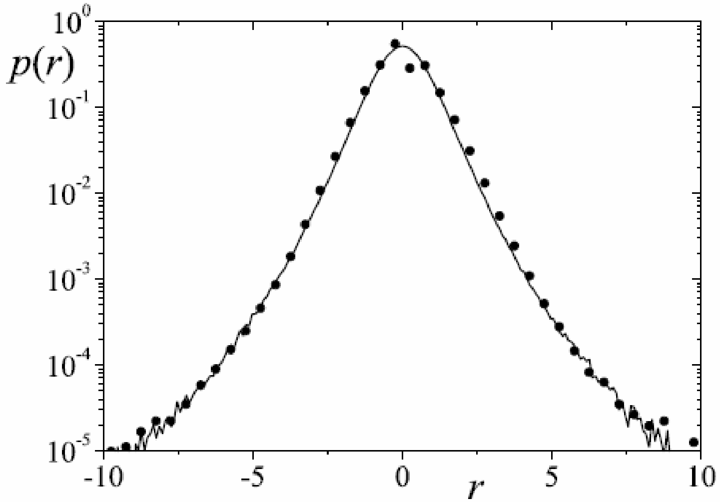
Model for price changes:

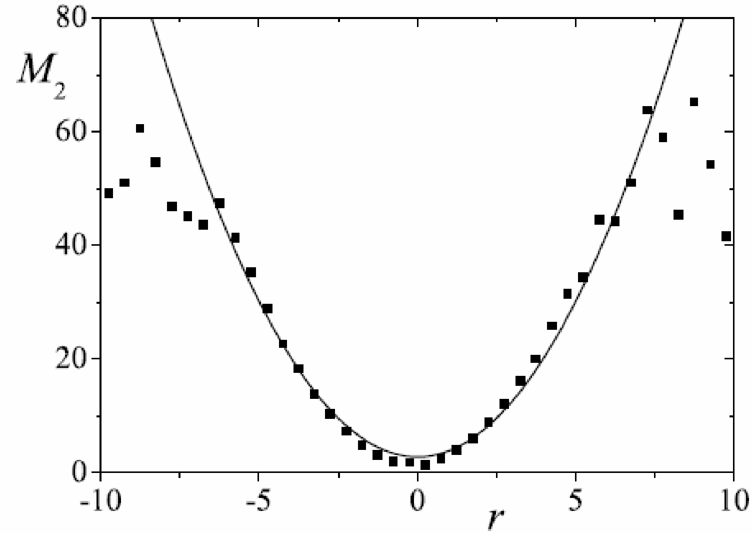
$$dr = -k\, r\, dt + \sqrt{\theta\ [p\,(r,t)]^{(1-q)}}\, dW_t \qquad (q \geq 1)$$

$$\frac{\partial p(r,t)}{\partial t} = \frac{\partial}{\partial r}\,[k\, r\, p(r,t)] + \frac{1}{2}\frac{\partial^2}{\partial r^2}\left\{\theta\ [p\,(r,t)]^{(2-q)}\right\}$$

$$p\,(r) = \frac{1}{Z}\left[1-(1-q)\,\beta\, r^2\right]^{\frac{1}{1-q}}$$

S.M.D. Queiros, L.G. Moyano, J. de Souza and C. T.
Eur. Phys. J. B **55**, 161 (2007)





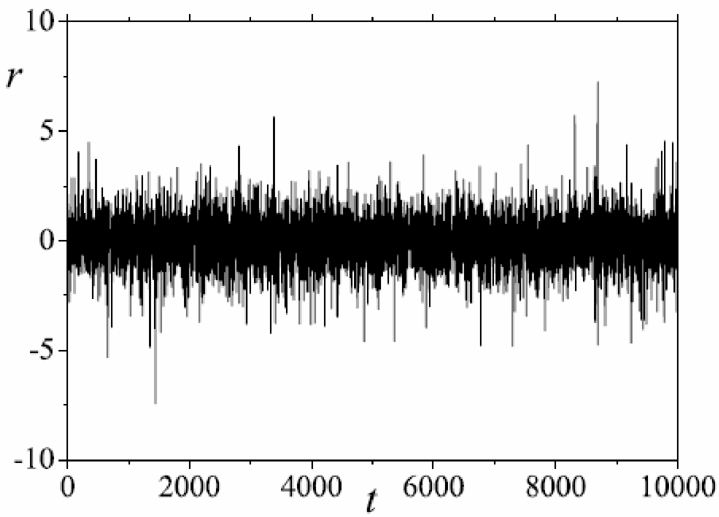


Fig. 1. Upper panel: probability density function vs. r. Symbols correspond to an average over the 30 equities used to built DJ30 and the line represents the PDF obtained from a time series generated by equation (15) (following the the procedure presented in Ref. [18]) which is presented on middle panel. Lower panel: 2nd Kramers-Moyal moment $M_2 \approx \tau\,\theta\,[p(r)]^{(1-q)} = \tau \frac{k}{2-q}\left[(5-3\,q)\,\sigma^2 + (q-1)\,r^2\right]$ from which k parameter is obtained and where the stationary hypothesis is assumed ($t_0 = -\infty \ll -k^{-1} \ll 0$). Parameter values: $\tau = 1\,\mathrm{min}$, $k = 2.40 \pm 0.04$, $\sigma = 0.930 \pm 0.08$ and $q = 1.31 \pm 0.02$. The points have been obtained from real data and the time scale is absolute.

S.M.D. Queiros, L.G. Moyano, J. de Souza and C. T.
Eur. Phys. J. B **55**, 161 (2007)

Model for traded volumes:

$$P(v)=\frac{1}{Z}\left(\frac{v}{\varphi}\right)^{\rho}\exp_q\left(-\frac{v}{\varphi}\right)$$

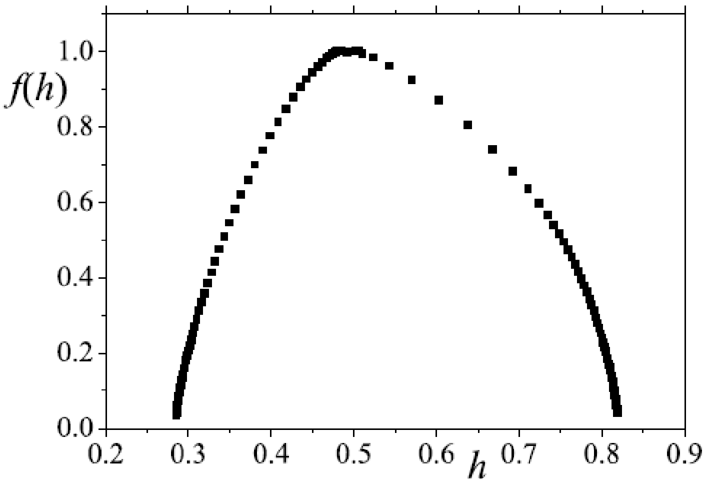


Fig. 4. Multi-fractal spectrum $f(h)$ vs. h for 1 min return averaged over the 30 equities with $h_{\min} = 0.28 \pm 0.04$ and $h_{\max} = 0.83 \pm 0.04$.

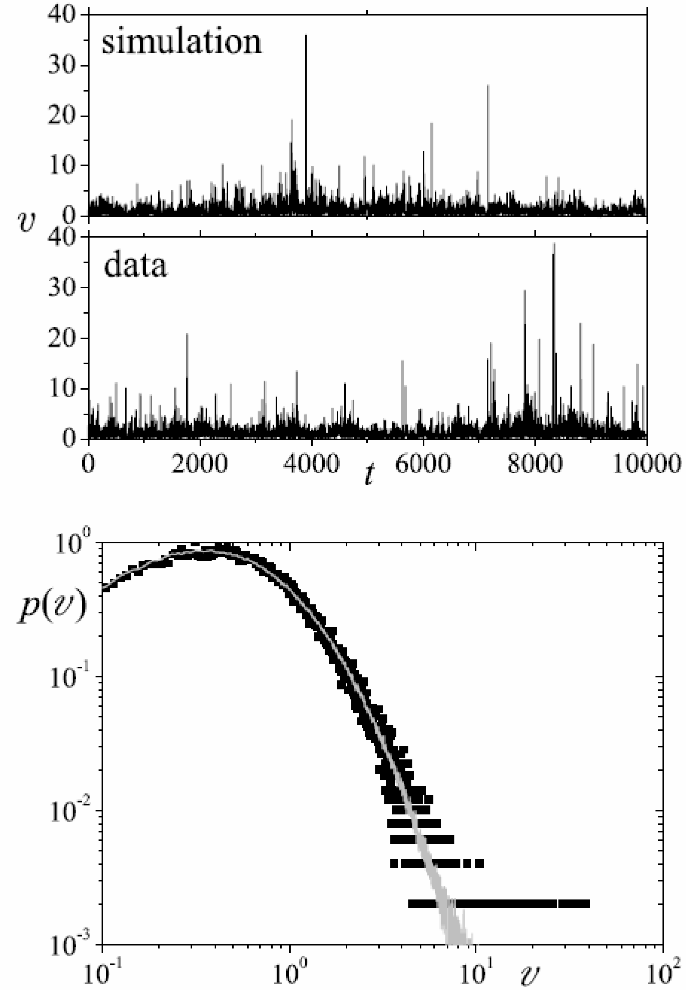


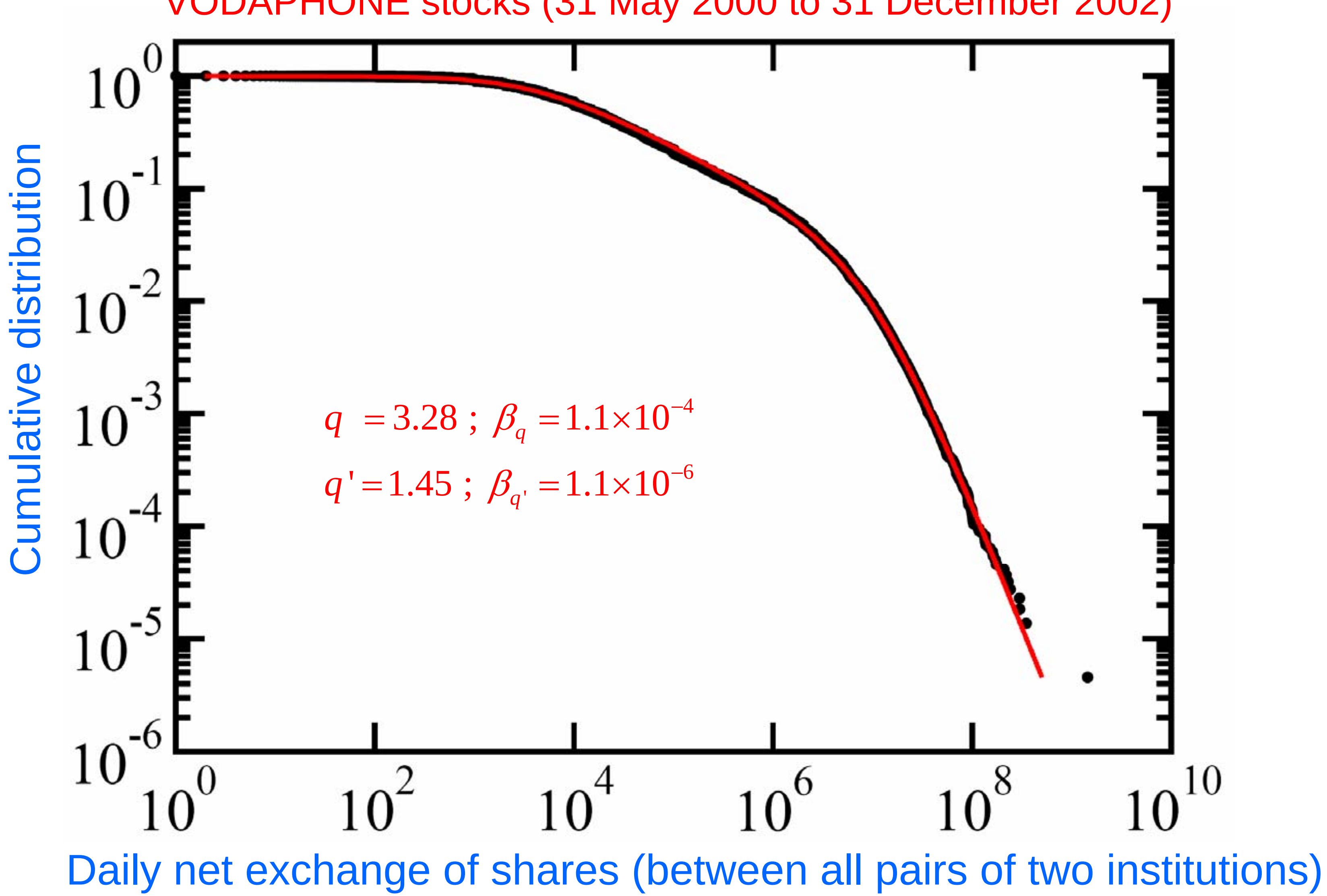
Fig. 3. Upper panel: excerpt of the time series generated by our dynamical mechanism (simulation) to replicate 1 min traded volume of Citigroup stocks at NYSE (data). Lower panel: 1 min traded volume of Citigroup stocks probability density function *vs.* traded volume. Symbols are for data, and solid line for the replica. Parameter values: $\theta = 0.212 \pm 0.003$, $\rho = 1.35 \pm 0.02$, and $q = 1.15 \pm 0.02$ ($\chi^2 = 3.6 \times 10^{-4}$, $R^2 = 0.994$).

S.M.D. Queiros, L.G. Moyano, J. de Souza and C. T.
Eur. Phys. J. B **55**, 161 (2007)

<u>LONDON STOCK EXCHANGE (Block market):</u>

Data: I.I. Zovko; **Fitting**: E.P. Borges (2005)

VODAPHONE stocks (31 May 2000 to 31 December 2002)



FORECASTING

S.M.D. Queiros and C. T.

Question:

Tomorrow, the price of this stock will increase or decrease?

(up/down question)

If no knowledge at all, the success rate should be 50%

Success rate in daily forecasts for stocks traded in BOVESPA (up/down)

Globo-Cabo	61.0	%
1 month interest-rate	61.5	%
2 months interest-rate	62.4	%
3 months interest-rate	61.8	%
6 months interest-rate	63.4	%
12 months interest-rate	63.5	%

Success rate in daily forecasts for stocks traded in London exchange from 1997 up to 2005 (up/down)

	Residual at 8 days
Barclays	55.0 %
British American Tobacco	55.2 %
BG Group	56.8 %
BP	55.8 %
BT Group	56.6 %
GlaxoSmithKline	54.3 %
Ladbrokes	52.2 %
Hanson	53.6 %
HSBC Holding	53.9 %
Lloyds	56.0 %

Success rate in daily forecasts for stocks traded in American markets from 1997 up to 2005 (up/down)

	Pricing	Residual at 8 days
American International Group (AIG)	54.1 %	53.1 %
Cisco Systems	54.8 %	52.9 %
General Electric	54.2 %	53.4 %
Intel	54.4 %	53.6 %
Lucent Technologies	54.7 %	53.0%
Microsoft	54.0 %	53.3%
Pfizer	54.4 %	53.0 %
SP500 (indice)	53.9 %	–
Time Warner	54.1 %	53.1 %
Wal Mart	54.0 %	53.3 %
Exxon Mobil	55.3 %	54.7%

Question:

Tomorrow, the price of this stock will increase or decrease? How much?

(8-interval question)

If no knowledge at all, the success rate should be 12.5 %

Success rate in forecasts for stocks traded in London exchange from 1997 up to 2005 (4 intervals up and 4 intervals down)

Residuals with magnitude greater than 3 residuals standard variation
Residuals with magnitude between 1 and 3 residuals standard variation
Residuals with magnitude between 1/2 and 1 standard variation
Residuals with magnitude between 0 and 1/2 standard variation

	Residual at 8 days
BG Group	16.7 %
BP	17.3 %
BT Group	15.7 %
GlaxoSmithKline	16.9 %
Ladbrokes	16.8 %
Hanson	18.7 %
Marks & Spencer	19.3 %

S.M.D. Queiros and C. T.

S.M.D. Queiros and C. T. versus Competitor / USA:

UK residual predictions for 1128 consecutive trading days (15 Dec 2000 to 7-Jun-2005)

Ticker	Prediction strength (QT)	Prediction strength (Competitor)
BARCLAY' BANK	0.100	0.260
BRITISH AMERICAN TOBACCO	0.089	0.082
BG GROUP	0.109	0.211
BRITISH PETROLEUM	0.117	0.025
BRITISH TELECOM	0.134	0.075
GLAXO SMITH-KLINE	0.079	0.060
HILTON GROUP	0.104	0.083
HANSON	0.076	0.050
LLOYD'S	0.121	0.094

FACIAL EXPRESSION RECOGNITION USING ADVANCED LOCAL BINARY PATTERNS, TSALLIS ENTROPIES AND GLOBAL APPEARANCE FEATURES

Shu Liao[1,2], Wei Fan[2], Albert C. S. Chung[1,2] and Dit-Yan Yeung[2]

[1]Lo Kwee-Seong Medical Image Analysis Laboratory
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The Hong Kong University of Science and Technology, Hong Kong.

Angry **Disgust** **Fear** **Happy** **Neutral** **Sadness** **Surprise**

Fig. 4. *Some sample images from the JAFFE database*

[2006 IEEE International Conference on Image Processing, pages 665 – 668]

Features	Classification Accuracy %
AMGFR [15]	82.46
LBP [6]	85.57
ALBP	88.26
Tsallis	85.36
ALBP + Tsallis	91.89
ALBP + Tsallis + NLDAI	94.59

Table 2. *Performance comparison of different approaches with resolution level 64×64 for the images from the JAFFE database*

Features	Classification accuracy (%) 48×48	32×32	16×16
AMGFR [15]	78.13	67.83	56.35
LBP [6]	81.44	77.28	68.02
ALBP	84.27	82.74	75.39
Tsallis	79.25	71.04	63.81
ALBP + Tsallis	87.31	85.73	80.40
ALBP + Tsallis + NLDAI	90.54	88.82	84.62

Table 3. *Performance comparison of different approaches with resolution levels 48×48, 32×32 and 16×16 for the images from the JAFFE database*

OBRIGADO